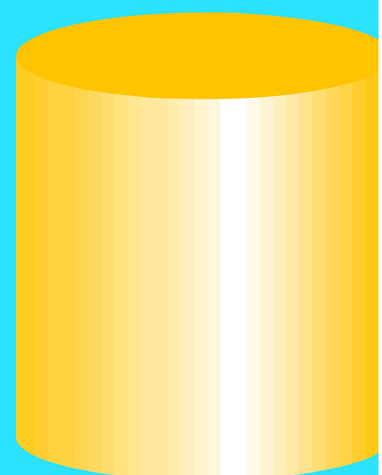
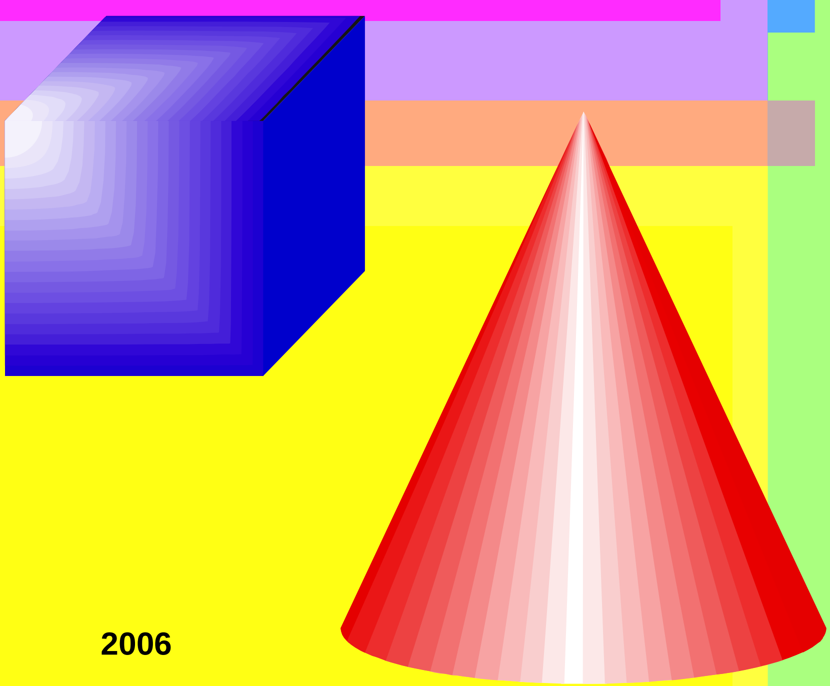
Teaching Shape and Space Concepts

- 2-D Shapes and 3-D Objects
- Transformations

Grades K–3





2006

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Questions or concerns regarding this document can be addressed to Debbie Duvall, Learning and Teaching Resources Branch, Alberta Education. Telephone 780–427–2984. To be connected toll free inside Alberta, dial 310–0000.

Several Web sites are listed in this document. These sites are listed as a service only to identify potentially useful ideas for teaching and learning. Alberta Education is not responsible for maintaining these external sites, nor does the listing of these sites constitute or imply endorsement of their content. The responsibility to evaluate these sites rests with the user.
Note: All Web site addresses were confirmed as accurate at the time of publication but are subject to change.

Acknowledgements

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Associate Professor Marj Horne, Australian Catholic University, writer
Karen Egge, Northern Regional Learning Consortium
Debbie Duvall, Learning and Teaching Resources Branch, Alberta Education
Greg Bishop, Assistant Director, Learning and Teaching Resources Branch, Alberta Education

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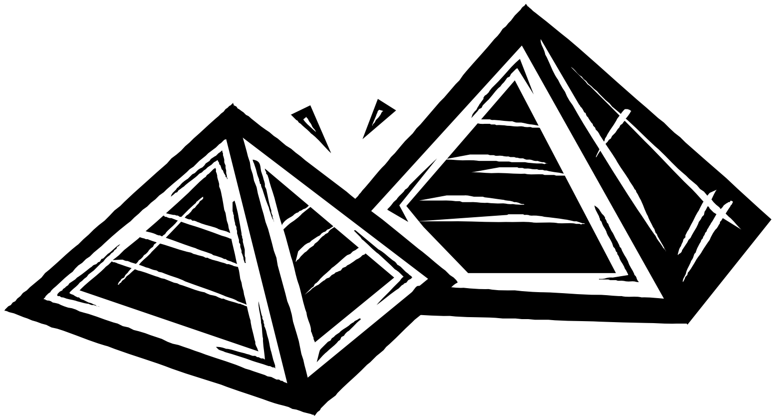
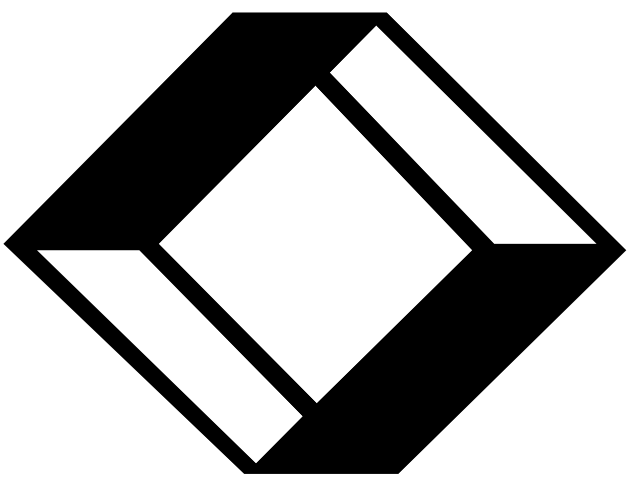
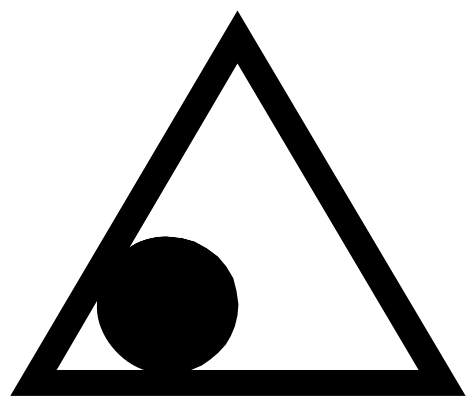






Outline for Professional Development
Shape and Space
Grades K–3

Welcome and Introductions Slide 1

Architecture provides a rich context for the study of Shape and Space. The background to the introductory slide is Federation Square in Melbourne, Victoria, Australia. The buildings in the square are all based on interesting tessellations of a right-angled triangle with a height twice the length of the base. This shape tessellates in many different ways, forming many other geometric shapes.

Warm-up Activity

What is my shape *– see activity booklet*

Following the activity, after the participants have sorted themselves into some different pairs and groups, discussed possibilities, and returned to their seats, introduce the record book.

Using each of the headings, ask them to fill in their notes as a record of the activity. Ask each question, allowing the group to discuss and answer. A few sample answers are given here.

What are the mathematics? Some responses might include the properties and the names of the shapes. They might also include the problem-solving aspects, such as being organized. The use and understanding of the language of shape is definitely one of the areas that is part of this activity. The activity requires communication and also visualization.

What do we want children to notice? Some answers might include ways of asking questions, language, types of information about shapes that helps us know what they are and the properties of the shapes.

What is the teacher's role and what questioning might the teacher do? The teacher may choose to be a participant, thus modelling questioning for the children. The teacher needs to listen carefully to the types of questions being asked and observe where difficulties occur so that attention can be drawn to useful practices in discussion. Two very useful questions are "What do you now know?" and "How do you know?" The role is more one of facilitation and data collection.

What are the features of the activity? One aspect of this activity is the students learning to ask questions which require a yes/no answer. It is a problem-solving task for each child which requires language use to find the solution. Once the children have found their mystery shape, however, the task continues with sorting and classification activity.

How can the activity be adapted / extended / made simpler in order to meet the needs of all students in the class? The sample shapes given can be varied to suit the needs of the group. 3-D is more challenging than 2-D, though having actual 3-D objects on display to support the pictures of the objects on the labels makes a difference.

Another version of this task is Celebrity Heads, which is also outlined in the activities. The difference is three children come to the front and have a card affixed to their heads with the shape or the name of a shape. They then ask the class questions to try to find out what is on their foreheads.

There is a recording sheet like this for each of the activities that will be done during the Professional Development.

Program for the Day Slides 2, 3

From the outline of the program, a number of activities are shown. There will be other information interspersed with the activities. Throughout the day, you are reminded that you have the recording sheets for your notes about activities.

The next slide shows the main Curriculum Outcomes for the What Is My Shape activity. Note the general curriculum aspects C, CN, PS, R and V. These are the critical aspects of Communication, Making Connections, Problem Solving, Reasoning and Visualization. While this activity does not completely meet all aspects of the outcomes, it requires participants to use the language, to reason, to visualize, to solve problems and to make connections. Build this on the ideas the participants raised when they completed the record sheet. Slide 4

Activity

Prototypes *– see activity booklet*

Draw a triangle. Draw a hexagon. Slides 5, 6

Discuss the key issues of the drawings. Ask the questions about triangles. Slides 7

Raise the issue of prototypical. Slide 8

Discuss the image children (and adults) have of particular shapes, returning to the idea of prototype. Discuss children in a research project deciding on calling a shape a half triangle because two together made a triangle. Slide 9, 10

Hand out the triangle sheet from the Early Numeracy Research Project (ENRP) interview and ask participants to decide on the numbers of the triangles. The slide has some similar examples. Slide 11

Ask participants to share their decisions in small groups. *What might children say when facing this task?*

Discuss why or why not each shape is or is not a triangle. The discussion should be about the key points of definition of a triangle. Key points are listed on the activity sheet.

Questions about hexagon. Slide 12

Show a non-prototypical hexagon. Slide 13

Discuss the language of the names and their meaning—hexagon, regular, classes, that square is a special quadrilateral. Discuss the importance of giving children a variety of experiences with different shapes. Slides 14, 15, 16

Activity

Making shapes from triangles *– see activity booklet*

If the issue of diamond does not arise, mention it as an important issue. Many texts refer to diamond as a shape in the same way they refer to triangle, but the two are very different.

Follow this activity with a discussion of properties contrasted with attributes. Slide 17

We use attribute to refer to characteristics but this includes aspects such as colour, size, shape, thickness and surface characteristics. Properties have special geometric meanings and are associated with aspects such as side lengths, parallel sides, equal angles, types of angles, symmetry and diagonal properties (e.g., diagonals of equal length, diagonals bisecting each other or the angle).

Shapes are classified according to their properties. The geometric shapes form a hierarchy that is inclusive so shapes can have more than one name. Just as I drive a car which is a sedan, a General Motors car, a Chevrolet, and a Corvette, so a rectangle is also a parallelogram and a quadrilateral and may be other things as well. Slides 18

Quadrilaterals Slide 19

It looks like a square, but what is it and how do we know? Slides 20, 21

This discussion is really focusing on background information rather than information for the students. The Making Shapes activity will create not only triangles and quadrilaterals such as a square, rectangle, parallelogram and trapezoid, but also a pentagon and some different hexagons. The next slide shows the curriculum outcomes for the activity. Allow participants time to fill in their record sheet, then choose one aspect such as the Teacher Role and Questioning to discuss. Slide 22

MORNING BREAK

Activity

Feely box *– see activity booklet*

Start this activity by showing the curriculum outcomes. Slide 23

This activity will require a volunteer to put his or her hands in the box and answer questions from the group. Run the Activity 1. Slide 24

Ask the person who had his or her hands in the box to describe how he or she did it and what he or she had to do.

Show the grid and discuss. The grid came from some work by Alan Bishop. In experiencing shape, children need to move between the three different forms: actual objects, representations of the object (e.g., diagrams or pictures) and language to describe the object (e.g., the name of the shape or its properties). Slide 25

Visualization is important. It is also the mediator between the three different forms described above. The person with their hands in the box moves from object to language via visualization. The others in the room move from language to representation, in this case a drawing, again via visualization.

Ask the tables to invent variations of the feely box for different levels and different purposes.

A variation for Kindergarten and Grade 1 is described in Activity 2.

A 3-D version is more of a challenge and meets different objectives. It is described in Activity 3.

This activity can be used at any level and the outcomes are dependent on the rules given for the particular task, the language and the nature of the shapes used. It has been used from Kindergarten classes through to Year 10, with slightly different sets of rules.

The mathematics here focuses particularly on the properties of shape and defining shapes by properties.

Van Hiele's levels are presented on the next slides. Slides 26, 27, 28

Discuss how the first years of school fit with Van Hiele's levels and link to the curriculum.

The findings from the ENRP in relation to the Properties of Shape are shown on the next slides, first with the Growth Points, then showing the achievement across grades K–2. Explain the graph by looking at a vertical line across the Grade 1 Nov 00 mark. This represents the students' understanding at the end of Grade 1. It shows a small number (about six percent) who are still showing no apparent recognition of shape names. About 12 percent recognized shapes holistically, basically at Van Hiele's Level 1. About 30 percent classified shapes attending to some visual features and thus could sort and compare shapes. About 40 percent could identify the classes and the remaining 12 percent could correctly identify the triangles knowing the defining features such as straight sides, "no gaps," etc. Slides 29, 30, 31

Groups start making a list of what they see as the big ideas of space raised so far. Slide 32

Activity

Create a copy *– see activity booklet*

Discuss degree of difficulty and orientation.

Ask participants to do the activity where one person on the table makes a shape and everyone else copies it. Did the copies all have the same orientation?

Ask participants to complete the record sheet, then, following some sharing of their responses, conclude the activity by showing the curriculum outcomes. Slide 33

Frames of reference make a difference. Some people see the shape through the eyes of the maker – others see it from their own position.

First, create a shape on the overhead projector without the light on. Call attention and tell the participants you are going to show them a shape for five seconds (longer with children). After you turn the light off, they are to reproduce the shape with the blocks. Ask participants to share the visualization techniques used.

For older classes or for the teachers at a workshop, creating a copy that has been reflected or rotated can meet other objectives at levels 4 and 5 and provide good background experiences for younger children. Many teachers will find it quite a challenge, particularly to create a copy that has been rotated 90^{o} clockwise.

Other extensions include creating a copy of a 3-D construction. This is done in the same way by using two, three or four blocks of a range of shapes.

Finally, extensions to drawing add another dimension. Show a simple drawing on the overhead projector for five seconds and ask participants to draw it.

Another way to work with this is for one participant to sit with their back to the group and to describe a drawing for the rest of the group to actually draw.

One important point to raise somewhere in the discussion is that Frames of reference make a difference. Some people see the shape through the eyes of the maker—others see it from their own position. Slide 34

LUNCH

Activity

Using technology tools *– see activities*

Cabri can be used at many levels in ways that achieve many outcomes. Demonstrate the drawing of shapes and the dynamic alteration of them.

Discuss the use of technology in explorations of space. Show the dynamic geometry slide and the one on using technology. Slides 35, 36

Demonstrate the different draw tools in Microsoft Word.

Ask participants to share some of the things they have done.

See curriculum outcomes Slide 37

Questioning

One key aspect of teaching is questioning. We structure an environment for children, then ask questions that help them attend to particular aspects so that they construct knowledge.

Participants look at p. 116 of the article:

Copley, J. V. (2000). "Geometry and spatial sense in the early childhood curriculum," in J. V. Copley, *The young child and mathematics*. (pp. 105 – 124) Washington, DC.: National Association for the Education of Young Children.

Ask participants to look at the list of questions on p. 116 and in their group choose three good types of questions with reasons explaining why they are good and why they like them.

Activity

Exploring shape and objects *– see activity booklet*

Sets of materials like this have many uses in a classroom. They provide a rich source of opportunities for sorting and counting, are interesting to students, and also enable classroom learning to closely connect with other environments.

The task specifically focuses on the ideas of attributes and noticing things that are the same about different objects. It also provides opportunity for language associated with shape to be developed.

Ask participants to complete the recording sheet. Show curriculum outcomes Slide 38

There is a need in the classroom for activities like this that have the specific purpose of encouraging children to notice characteristics and to use language, but also allowing them to choose.

In looking at the objects, and indeed, in looking at the environment around them, young children are learning to visualize shape. Can they actually see the circle in the can of food? Do they see the rectangle when they look at a door? Can they see that the door has essentially the same shape as the cover of a book?

Other aspects of visualization include the ability to see shapes in different orientations in an environment. What can you see in the room around that is the same shape as the shape on this slide. Ask for more until there have been a good range so that you can indicate those where the respondent has visually rotated the shape. Slide 39

Activity

Shapes around us *– see activity booklet*

This activity is about making connections between 2-D and 3-D and raising students' awareness of their environment and shapes in the world. The slides here are taken in the surrounds of a school. It is a good idea to try to take a few pictures in the surrounds of the professional development day to include instead of or as well as these slides.

It is important to stress again the difference between a triangular shape and a triangle.

The slides in order are of a drinking fountain, a seat in the playground, a bolt on play equipment, hexagonal shapes on an adventure climbing frame, a rubber mat at a door for wiping boots and the facia of a gable. Slides 41–46

The next set of slides are black and white photographs of shapes around the venue where the Professional Development was run in Edmonton. If possible, it is a good idea to take your own digital photographs around the venue and use them instead. Slides 47–63

The ENRP looked at growth points for the development of visualization. These were based on the research literature at the time and focused on visualizing shapes statically, then rotating and reflecting them mentally, followed by dynamic changing of shapes. Children today may see this dynamic in the stretching of pictures and morphing on computers. Slides 64, 65

Activity

Straw and string constructions *– see activity booklet*

While this activity is mainly 3-D construction, first looking at the 2-D shapes and observing practical properties such as the immobility of a triangle when compared to a square is really important and helps make connections to such things as buildings and architecture.

After the activity, have participants fill in the record sheet. The curriculum outcomes are on the next slide. Slide 66

There are other possible construction media that can be used to build such framework models, and other construction material that can build net models. Slides 67, 68, 69

These next slides show a Grade 5 group who have constructed polydehra and, in particular, geodomes, with a kit of dowel rods and plastic tube joints. Slides 70, 71

One aspect of the 3-D shapes is the Platonic solids – the only five fully regular 3-D polyhedra. Slide 72

Activity

Reflection cube *– see activity booklet*

Reflect on the day and what we have done together.

Complete the list of big ideas that was started this morning. Slide 73

Choose the six most important ones. Decide which net will form a cube and write these six ideas on the faces of the cube.

Make the cube.

The task initially requires visualization to decide which shape would fold into a cube. The main purpose, though, is reflection on the day's discussions.

Having students select a net and make a cube can also be used as a reflection for them and geometrically relates to the curriculum outcomes shown on the next slide. Slide 74

Continuing the Journey

No professional development should conclude without participants having time to reflect and think about "where to next." Allow time for discussion of this critical aspect and ask everyone to commit themselves to doing one thing as a result of today's session.

Materials Needed for the Professional Development Day Grades K–3

Cardboard box with holes cut and flaps attached for a Feely Box.

Collection of simple-shaped household and classroom objects such as cans, small boxes (including a Toblerone box or others of unusual shapes), dice, attribute blocks, lids, buttons, cubes, building blocks, cardboard rolls

Computer with PowerPoint (optional) or Geometer Sketchpad and Microsoft Word

Fine cotton string – found in supermarkets for tying poultry

Glue

Newspaper

Origami squares (kinder squares) of two different colours – at least one of each colour for each participant

Overhead projector

Pattern blocks – at least two of each for each participant

Pencils or pens

Pictures from environment

Scissors – at least one pair between two

Adhesive tape

Scrap paper for writing

Set of 2-D shapes – samples are shown in the Feely Box Activity

Set of sticky labels printed as per the sample sheets in What Is My Shape? – enough for at least one for each participant.

Sheets of nets of cubes – at least one sheet for each participant

Straws – at least 10 for each participant

Two sets of 3-D shapes, one large. The set should include a cone, a cylinder and a sphere as well as prisms and pyramids

Checklist of Outcomes Addressed in the Professional Development

While many activities are rich and open to inclusion of more outcomes, the key outcomes for each activity are listed here.

		What is My Shape?	3-D Celebrity Heads	Making Shapes	Feely Box Geometry	Create a Copy	Cabri Geometry	Exploring Shape & Objects	Shapes Around Us	String & Straw Construction	Reflection Cube
K.2	Sort 3-D objects using a single attribute.							X			
K.3	Build and describe 3-D objects.							X			
1.2	Sort 2-D shapes using one attribute and explain the sorting rule.			X			X				
1.3	Replicate composite 2-D shapes and 3-D objects.			X		X		X			
1.4	Compare 2-D shapes to parts of 3-D objects in the environment.									X	
2.6	Sort 2-D shapes using two attributes and explain the sorting rule.			X							
2.7	Describe, compare and construct 3-D objects.		X		X				X		X
2.8	Describe, compare and construct 2-D shapes including triangles, squares, rectangles and circles.	X	X	X	X	X	X				
2.9	Identify 2-D shapes as parts of 3-D objects in the environment.									X	
3.6	Describe 3-D objects according to the shape of their faces and the number of edges and vertices.		X		X				X		
3.7	Sort regular and irregular polygons including triangles, quadrilaterals, pentagons, hexagons and octagons, according to the number of sides.			X	X						

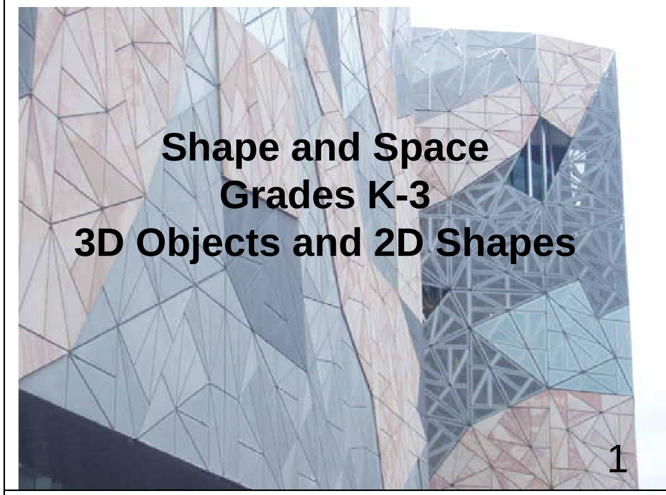
Background Papers

Clements, D. H. (1999). Geometric and spatial thinking in young children," in J. V. Copley (Ed.), *Mathematics in the early years* (pp. 178–191). Reston, VA: National Council of Teachers of Mathematics.

Copley, J. V. (2000). "Geometry and spatial sense," in *The young child and mathematics*. (pp. 105–124) Washington, DC: National Association for the Education of Young Children.

Doverborg, E., & Pramling Samuelsson, I. (2001). "Children's experience of shape in space." *For the Learning of Mathematics, 21*(3), 32–38.

Horne, M. (2003). "Properties of shape." *Australian Primary Mathematics Classroom*, *8*(2), 8–13.



Shape and Space
Grades K-3
3D Objects and 2D Shapes

1

Program for the Day

Welcome and Introduction
- What Is My Shape?
- Prototypes
- Making Shapes

Coffee
- Feely Box Geometry
- Create a Copy

Lunch

2

Program for the Day

- Cabri Geometry
- Visualization on a Geoboard
- Straw and String Construction
- Shapes Around Us
- Reflection Cube

Continuing the Journey

3

Curriculum Outcomes
What is My Shape?

2.8 Describe, compare and construct 2-D shapes including triangles, squares, rectangles and circles. [C, CN, R, V]

4

Please draw a triangle

5

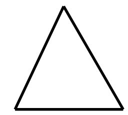
Please draw a hexagon

6

- Did your triangle have a horizontal line?
- Was the other vertex above the line?
- Was there a vertical line in your triangle?
- Was there a right angle in your triangle?
- Was your triangle close to being equilateral?
- Was your triangle almost isosceles?

7

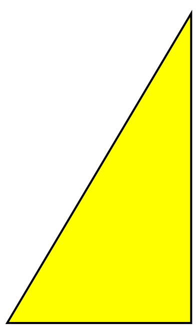
Did you draw the standard, prototypical triangle?
- standing on its base
- equilateral or isosceles



Did you think about other possibilities?

What are the properties of a triangle?

8



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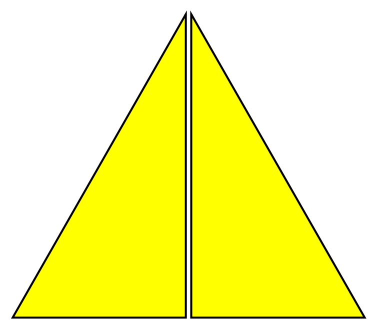
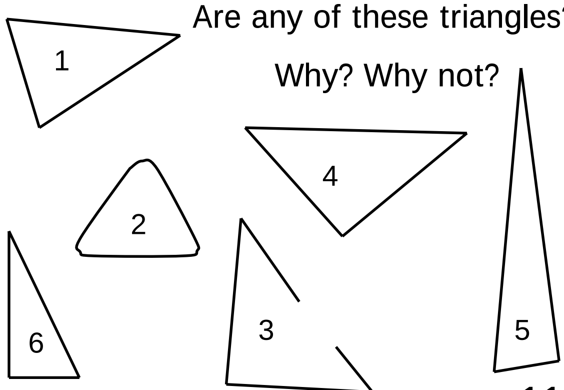


10

Are any of these triangles?

Why? Why not?



11

- Did your hexagon have a horizontal line?
- Was there a vertical line in your hexagon?
- Was your hexagon shaped like the beehive hexagon?
- Did your hexagon have a right angle in it?
- Did it have two right angles? Three?
- More than three?

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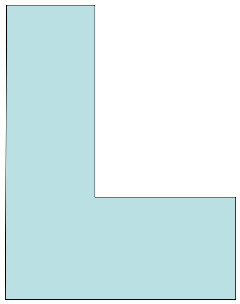
My Hexagon



13

Triangle		three angles
Quadrilateral	Latin	four sides
Pentagon	Greek	five angles/corners
Polygon	Greek	many angles/corners
Diagonal	Greek	between two corners

14

The word hexagon covers all polygons with six sides.

The word regular is used to specify a special shape in the class.

A square is a regular quadrilateral.

We do not need to refer to all other quadrilaterals as irregular.

Dichotomy—regular/irregular.

15

Students' perceptions of a shape are often based on a limited number of examples.

They are often unfamiliar with nonstandard examples (e.g., textbooks often draw shapes with horizontal baselines).

Students should have exposure to a large number of examples of the various shapes in the form of pictures, manipulatives and real-world situations.

Nonexamples should also be considered.

16

Properties

Attributes

17

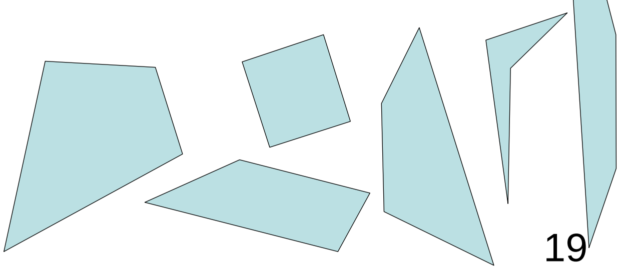
Classification of Shapes

- Classification (defined by properties)
- Hierarchy (shapes can have more than one name)

18

Quadrilaterals

- Planar (flat) closed (“no gaps, no extras”) shapes with four straight sides.



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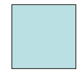

- has 4 straight sides, planar and closed so is a quadrilateral
- also has a pair of parallel sides so is at least a trapezium
- actually has two pairs of parallel sides so is a parallelogram
- has right angles so is a rectangle
- has all sides equal in length so is a square

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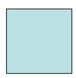

- has 4 straight sides, planar and closed so is a quadrilateral
- also has two pairs of adjacent sides equal in length so is a kite
- actually has all sides equal in length so is a rhombus
- has a right angle as well so is a square

21

Curriculum Outcomes Making Shapes

1.3 Replicate composite 2-D shapes. [CN, PS, V]

1.2 Sort 2-D shapes using one attribute, and explain the sorting rule. [C, CN, R, V]

2.6 Sort 2-D shapes using two attributes, and explain the sorting rule. [C, CN, R, V]

2.8 Describe compare and construct 2-D shapes. [C, CN, R, V]

3.7 Sort regular and irregular polygons according to the number of sides. [C, CN, R, V]

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Curriculum Outcomes Feely Box Geometry

2.7 Describe, compare and construct 3-D objects. [C, CN, R, V]

2.8 Describe, compare and construct 2-D shapes. [C, CN, R, V]

3.6 Describe 3-D objects according to the shape of their faces, and the number of edges and vertices. [C, CN, PS, R, V]

3.7 Sort regular and irregular polygons according to the number of sides. [C, CN, R, V]

23

Possible Answers

- Yes
- No
- I don’t understand the question. Please ask it in another way.
- I don’t know. Please tell me how I can find out.

24

Translations with Visualization as the Mediator

From / To	Object	Representation	Language
Object			
Represen-tation			
Language			

25

Van Hiele Levels

Level 1. **Visual**. Children at this level identify and operate on shapes according to appearance. They use the idea of congruency of visual properties and identification is based on these visual properties, such as “it is a cube because it looks like a box” or “it is a rectangle because it looks like a door.” At this stage, little attention is given to properties of the shapes. In research done in the United States, it was found that at least half the Grade 6 children were still operating at Level 1.

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Van Hiele Levels

Level 2. Descriptive/Analytic. At this level, the properties of shapes assume precedence with children characterizing shapes by their properties. The focus is on relationships within classes rather than relationships between. A cube is now a cube because it is three dimensional with all faces the same sized squares. Students should be operating at this level when they enter high school and at the early stages of it as a minimum. Only 44 percent of students in the US were operating consistently at Level 2 at Grade 9. Other studies have shown that 40 percent of students complete high school below Level 2.

27

Van Hiele Levels

Level 3. Abstract/Relational. Students are able to deal with abstract definitions, understanding the idea of necessary and sufficient conditions, recognizing a hierarchy, and reasoning about the properties of classes of figures. Logical argument is part of this level. Internationally most geometry curriculum strive to attain this level. It varies between countries and states as to whether students are encouraged to attain this level in school.

Clements, 1994, in Grouws, Handbook of Mathematics Education

28

Growth Points for Properties of Shape

0. Not apparent
   Not yet able to recognize and match simple shapes.

1. Holistic recognition of shape
   Can recognize resemblances and match some simple shapes, using standard “prototypes.”

2. Classification of shapes, attending to visual features
   Can sort and compare shapes, using some geometrical language to describe features.

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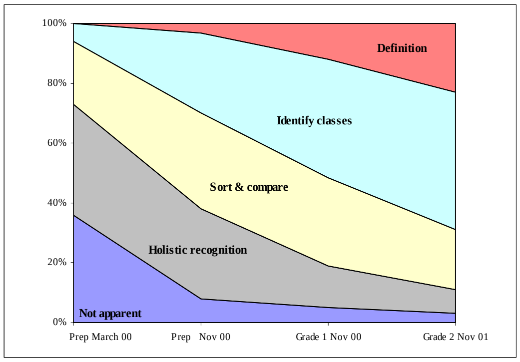
Growth Points for Properties of Shape

3. Identification of “classes of shapes” by some properties
   Uses properties of shapes to classify shapes into classes, using appropriate language.

4. Definition of shapes using properties
   States and understands conditions for defining key shapes.

30

Properties of Shape



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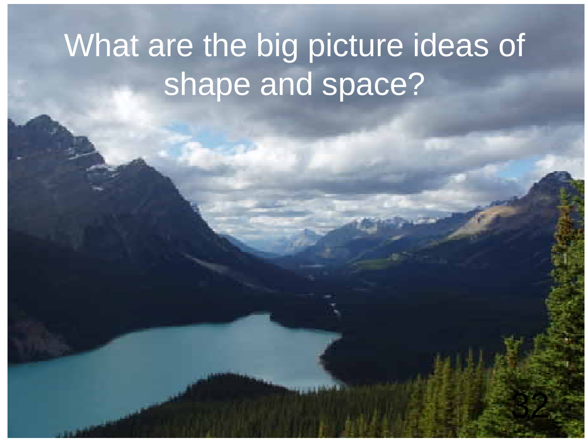

32

Curriculum Outcomes
Create a Copy

1.3 Replicate composite 2-D shapes and 3-D objects. [CN, PS, V]

33

Frames of Reference

How do you *see*?

From whose perspective do you visualize?

34

Dynamic Geometry Software

Cabri Geometry

- the original – designed in France at Grenoble University
- has been used from pre-school children to PhD students in mathematics

Geometer Sketchpad

- US designed
- used widely in schools, particularly in North America

35

Using Technology

- MS Word
- Cabri/Geometer Sketchpad
- Specific-focused software, such as Building Perspectives and Tesselmania

36

Curriculum Outcomes
Cabri Geometry

2.8 Describe, compare and construct 2-D shapes. [C, CN, R, V]

3.7 Sort regular and irregular polygons according to the number of sides. [C, CN, R, V]

37

Curriculum Outcomes
Exploring Shape and Objects

K.2 Sort 3-D objects using a single attribute. [C, CN, PS, R, V]

K.3 Build and describe 3-D objects. [CN, PS, V]

38

Visualization

Look around the room. Can you see something that is the same shape as the shape below?

Are there any more like this?

39

Curriculum Outcomes
Shapes Around Us

1.4 Compare 2-D shapes to parts of 3-D objects in the environment. [C, CN, V]

2.9 Identify 2-D shapes as parts of 3-D objects in the environment. [C, CN, R, V]

3.6 Describe 3-D objects according to the shape of their faces, and the number of edges and vertices. [C, CN, PS, R, V]

3.7 Sort regular and irregular polygons according to the number of sides. [C, CN, R, V]

40

Where is it? What is it?
What shapes are in it?

41

Where is it? What is it?
What shapes are in it?

42

Where is it? What is it?
What shapes are in it?

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Where is it? What is it?
What shapes are in it?

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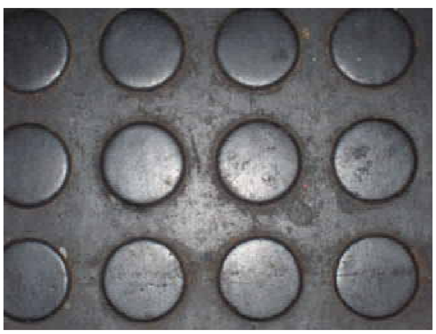
Where is it? What is it?
What shapes are in it?



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Where is it? What is it?
What shapes are in it?

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Where is it? What is it?
What shapes are in it?

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What shapes are in it?

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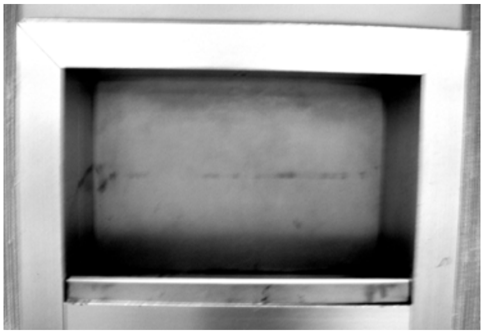
Where is it? What is it?
What shapes are in it?



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What shapes are in it?

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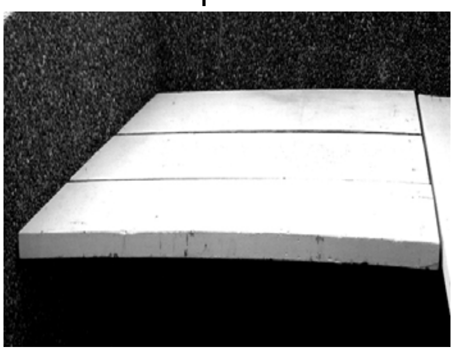
What shapes are in it?



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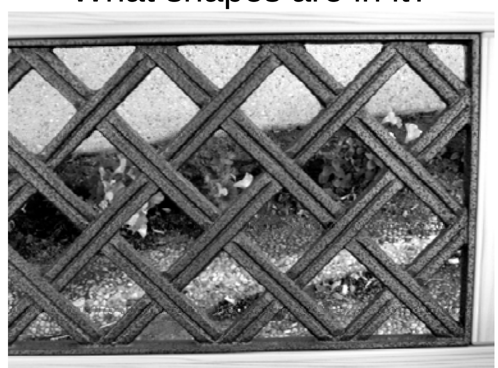
What shapes are in it?



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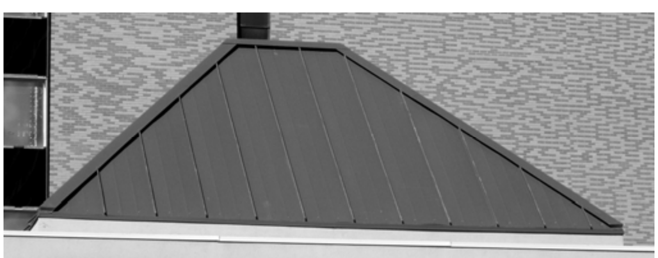
What shapes are in it?



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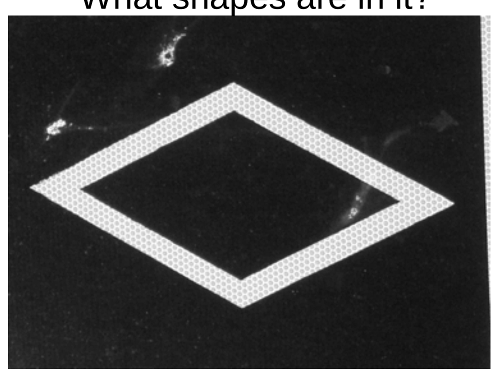
What shapes are in it?



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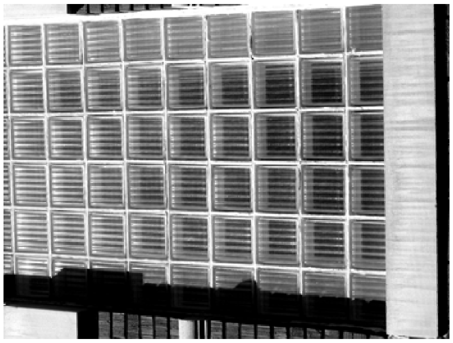
What shapes are in it?



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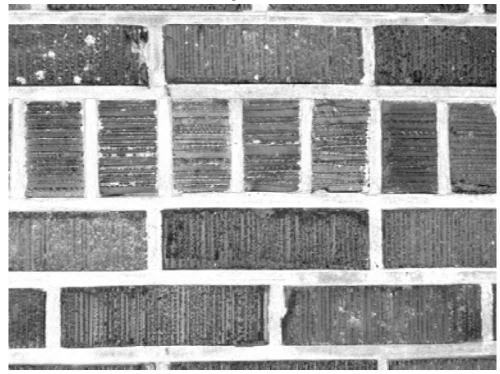
What shapes are in it?



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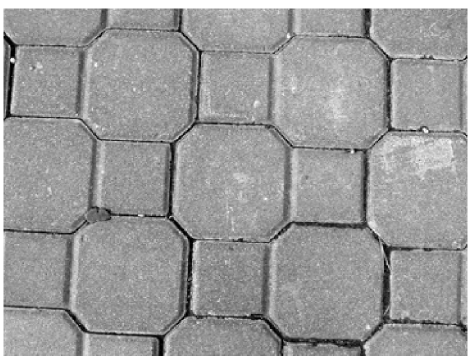
What shapes are in it?



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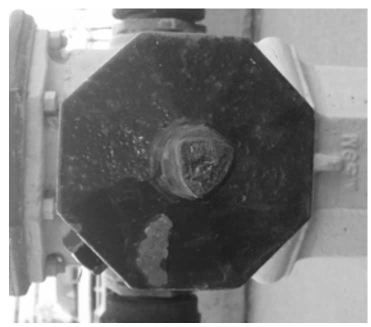
What shapes are in it?



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What shapes are in it?

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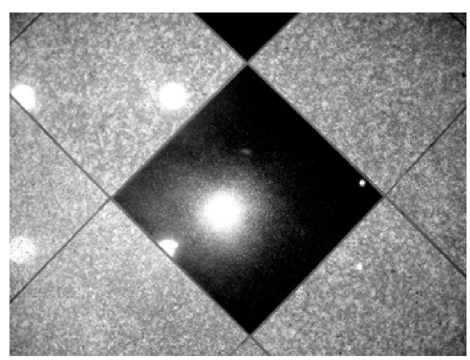
What shapes are in it?



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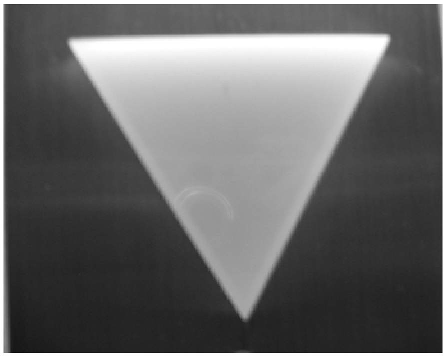
What shapes are in it?



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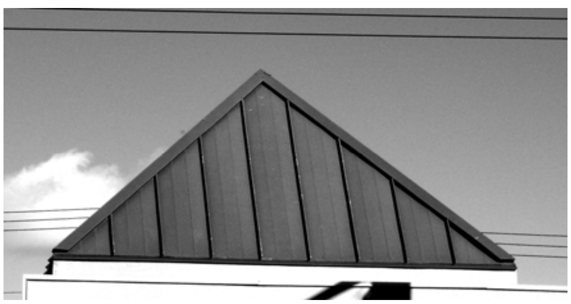
Where is it? What is it?
What shapes are in it?

62

Where is it? What is it?
What shapes are in it?



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Growth Points for Visualization

0. Not apparent
   Not yet able to visualize simple shapes.
1. Static, pictorial images formed in conjunction with models or manipulatives
   Able to recognize static images in embedded situations.
2. Reorientation of objects mentally
   Can visualize the effect of simple flipping, sliding and turning of objects.

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Growth Points for Visualization

3. Dynamic imagery
   Uses dynamic imagery to visualize manipulation of objects by transforming and rearranging.
4. Extending and applying visualization and orientation
   Can combine a range of visualization strategies in increasingly complex situations.

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Curriculum Outcomes
Straw and String Construction

2.7 Describe, compare and construct 3-D objects. [C, CN, R, V]

3.6 Describe 3-D objects according to the shape of the faces, and the number of edges and vertices. [C, CN, PS, R, V]

66

Construction

- Frame models such as straw and string or pipecleaners
- Solid materials such as play dough or modelling clay
- Net models such as geo-shapes

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68

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70

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Platonic Solids

All regular in all respects.

- Tetrahedron – four triangular faces
- Cube (hexahedron) – six square faces
- Octahedron – eight triangular faces
- Dodecahedron – twelve pentagonal faces
- Icosahedron – twenty triangular faces

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Curriculum Outcomes Reflection Cube

2.7 Describe, compare and construct 3-D objects. [C, CN, R, V]

74

Elementary Mathematics Workshop Feedback Form

Teaching Shape and Space Concepts: Grade K–3

What I liked best about the workshop:

What I would like to see changed in this workshop:

Other general comments:

Comments on specific lessons:

Name of lesson: ______________________________
Comments:____

Comments on specific lessons:

Name of lesson: ______________________________
Comments:____
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Comments on specific lessons:

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Comments on specific lessons:

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The last word is yours:

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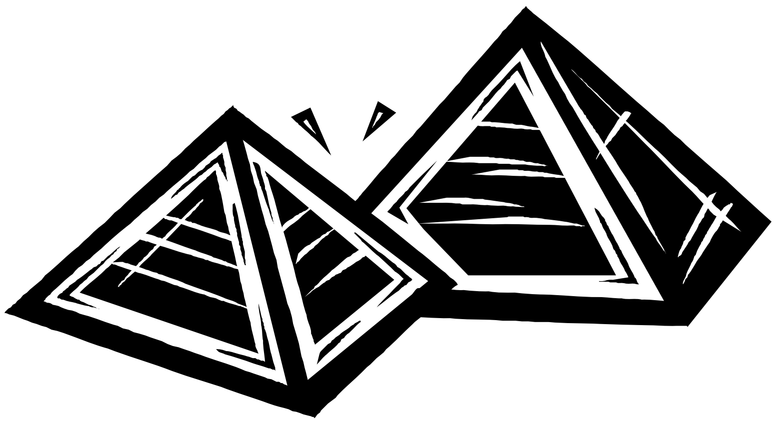
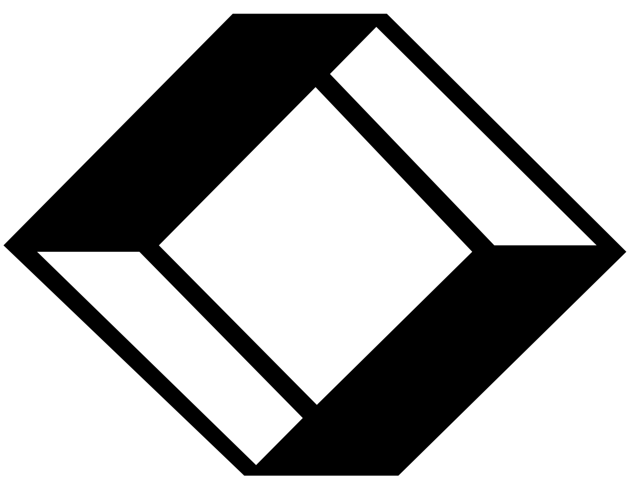
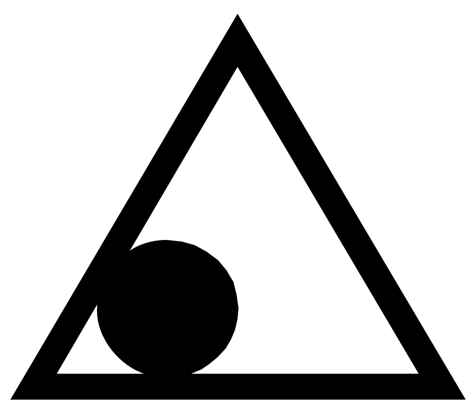
Name: (optional) ____

Contact Information: Telephone ___________________ **E-mail** ____________________________

Activities







What Is My Shape?

Materials:
- Set of sticky labels printed as per blackline masters (see pp. 34–35) or similar cut so each person can be given one.
- Sheet 2 blackline master includes pentagons, hexagons and octagons and may be used with older children or as a challenge for children who need extension.

Grade: 2

Learner Outcomes: Grade 2, No. 8
Describe, compare and construct 2-D shapes including triangles, squares, rectangles and circles.
[C, CN, R, V]

Activity Description: The purpose of this task is for the student to identify a 2-D shape by asking questions about it. This will require them to use language to ask about its properties. At the same time, other students are identifying shapes and answering questions about their properties. It strongly incorporates the critical mathematical components of communication, mathematical reasoning and visualization.

Background Information: Early recognition of shape is very dependent on prototypical images so "it is a triangle because it looks like one." This activity moves the children to thinking specifically about the properties of the shapes in the identification process.

The Activity:

1. Each student affixes his or her label (from either Sheet 1, Sheet 2 or similar) to the back of someone else.
2. The task is to identify the shape on the label on your own back by asking questions according to the following rules:
   - Each student may be asked only two questions (but you may ask many people). A student may ask: "Is it a triangle?" (no) "Is it a square?" (no), then the student has finished asking questions of that person and needs to ask another student more questions.
   - The questions must be such that the answer is "yes" or "no."
   - There are four possible answers to a question:
     - yes
     - no

- I don't understand – please ask it in another way
- I do not know how to answer that.

3. The first time the activity is run, allow any type of questions.

- Most young children will just ask questions such as "Is it a triangle?" Listen carefully to students' questions and identify any students who ask about properties.
- After the students have asked some questions, call everyone's attention and ask the students what types of questions they could ask. They might ask questions about the sides of the shape; e.g., "Does it have just three sides? Are all the sides straight?" They might ask questions about the angles or corners, e.g., "Does it have any corners? Does it have four corners?" These questions are about the properties of the shape and help in the identification.
- Students might also ask questions about the general image. These questions about the look (e.g., "Does it look like a door?") can assist young students in identification, but are not as efficient.
- The students see the shapes the other students have on their backs and hear other students asking questions that assist them to decide what questions to ask.

4. When the students have correctly identified the shapes, the labels are moved from their backs to their fronts.

- If there are students who seem to be having difficulty identifying their shape, join in the conversation, asking them what they have found out so far and what else they might be able to find out.
- Through this process, assist them to focus on the important details and to identify the shape.

5. Ask the students then to find a partner whose shape is like theirs in some way.

- Go around the group asking the students to say how their two shapes are alike and also what is different about them. This idea of focusing on what is the same and what is different is a really important technique in developing mathematical ideas.
- Ask students to share the questions that they asked, particularly any students you noted who did ask about properties.

6. Repeat the pairing activity, asking them to find a different student with whom to pair.

- Follow this by asking them to form groups of four to six where their group has shapes that are alike in some way and repeat the discussion about same and different.

7. Depending on the time, repeat the activity, using the same or similar labels, but this time tell them they are not to use the name of the shape in their questions until they are sure they know what their shape is. Of course some students will think they know what their shape is immediately.

- Listen for this as it gives you good information about the students' developing understanding of reasoning and properties of shapes.
- As before, if some students appear to be having difficulty deciding on what questions to ask after the activity has been running for a short time, stop the activity. Briefly brainstorm what types of questions are useful. After a number of questions have been suggested allow the students to return to the activity.

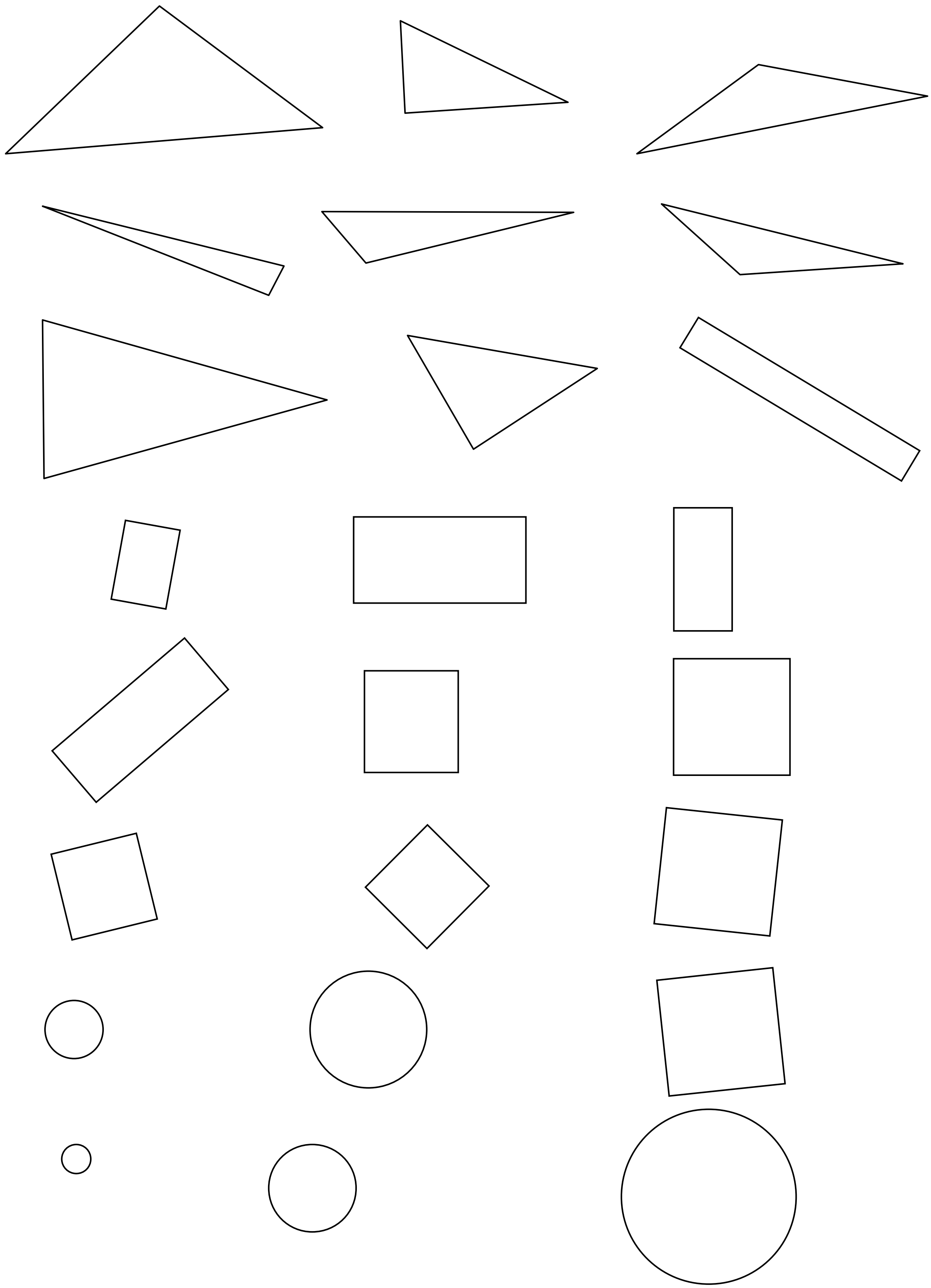
Again when a student has identified the shape, the label is moved to his or her front, but he or she still participates by answering questions.

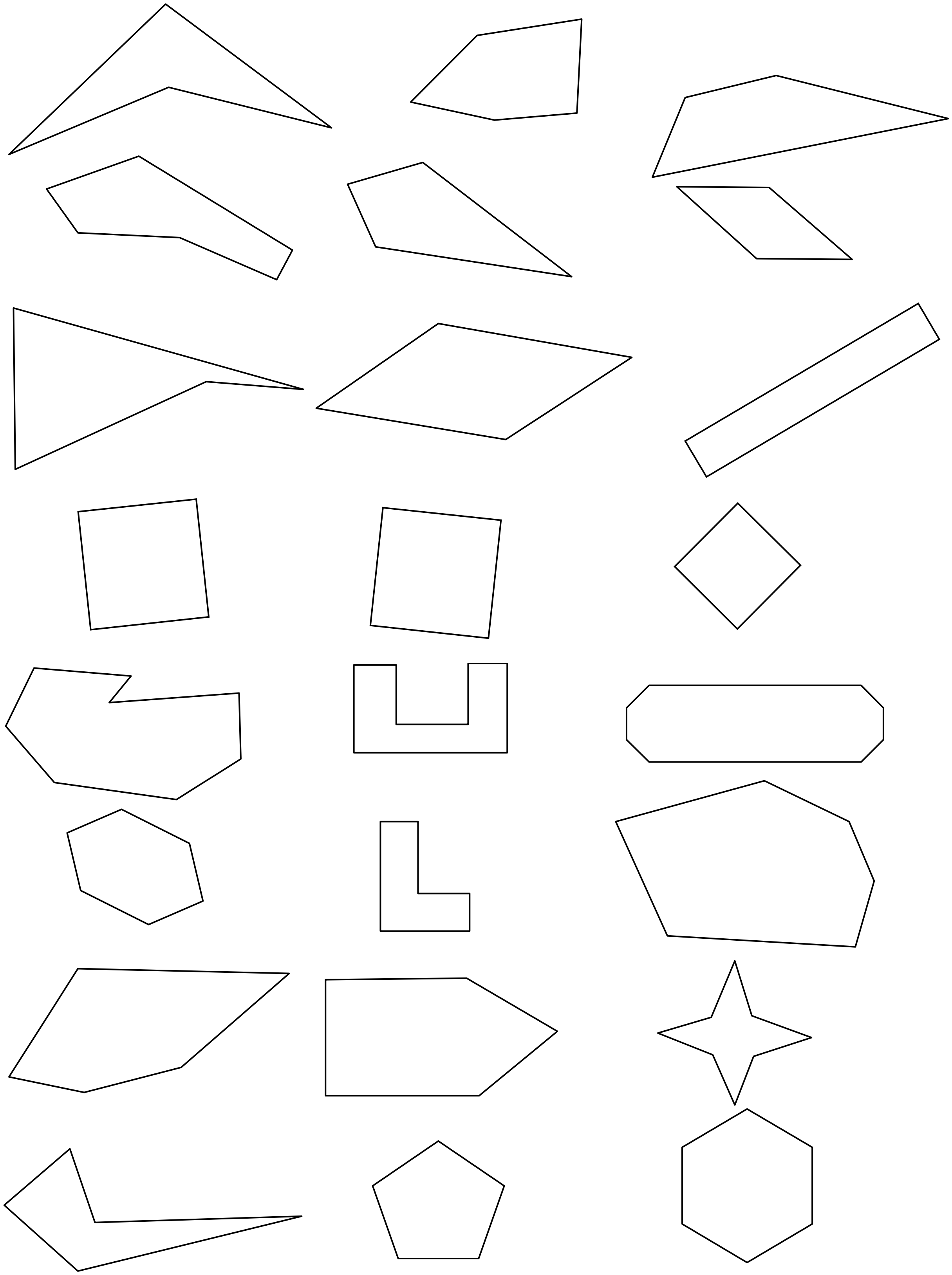
- Once students all have moved their labels, ask them to find a partner whose shape matches theirs in some way. Share the matches.
- Repeat, but this time suggest that they form groups using different reasons for matching. For example, in the pairing two students may have said, "Our shapes both have four sides." In their next pairing they might focus on something different such as the angles—"our shapes have square corners" (right angles, though the children may not know the language) or "our shapes are all quadrilaterals."

8. Finish the activity by consolidating the learning. You will have the whole class together as they share how they formed their groups and what was the same and what was different about the shapes. This began the consolidation process. Finish by going through some of the key shapes and asking questions to highlight the properties of the shape and how we know what the shape is.

- For example: "Show a quadrilateral or the group of quadrilaterals. What is it that means we know that they are quadrilaterals?"
- The key points are that quadrilaterals have four sides, the sides are all straight, the sides all join up (closed) and the shapes are flat (planar).
- The aspects of closed and planar may not arise because there are not four-sided figures that do not have these properties in the group.
- Different activities would be needed to highlight these properties. For example, these properties may arise in the Straw and String Construction activity.

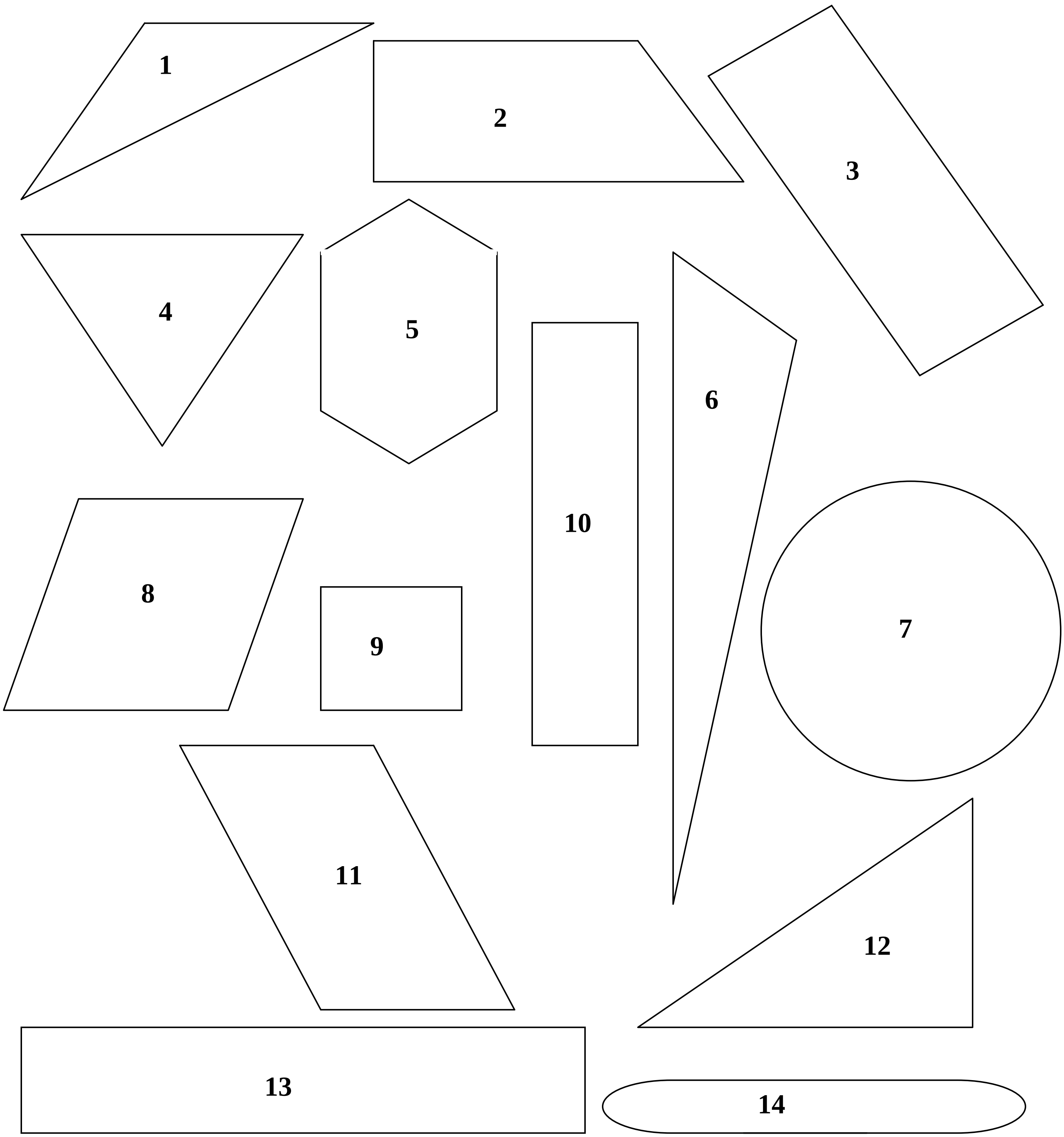
blackline master 1





Objective: Classifies 2-D figures: circles, squares, triangles and rectangles

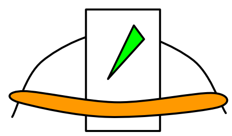
Glue these figures on cardboard and cut them out.



3-D Celebrity Heads

Materials:

- Set of pictures on cards that can stand on a simple elastic headband so each student playing can have one affixed to his or her head or forehead (see blackline masters pp. 39, 40 for examples).
- Elastic headbands.
- A set of large 3-D shapes to put on display which should include a cone, a sphere and a cylinder as well as a cube and other polyhedra such as a pyramid.



Grade: 2, 3

Learner Outcomes:

Grade 2, No. 8
Describe 2-D shapes including triangles, squares, rectangles and circles.
[C, CN, R, V]

Grade 3, No. 6
Describe 3-D objects according to the shape of their faces, and the number of edges and vertices.
[C, CN, PS, R, V]

Activity Description:

This is a variation of the What Is My shape? activity. The purpose is for the children to identify a 2-D shape or a picture of a 3-D shape by asking questions about it. This will require them to use language to ask about its properties. At the same time, other children are identifying shapes and answering questions about their properties. It strongly incorporates the critical mathematical components of communication, mathematical reasoning and visualization.

The Activity:

1. Three students come out and each has a label or card (unseen by them) affixed to his or her forehead.

2. They take turns to ask yes/no questions of the group to try to identify their shape. They are not to use the name of the shape in their question but must ask about its sides and/or angles.

- While angles aren't formally introduced until later, the students will ask questions about the corners of the shapes. For 3-D shapes, they can ask about the shapes of its faces or other properties such as faces, edges and vertices.
- Play with the 2-D cards first.

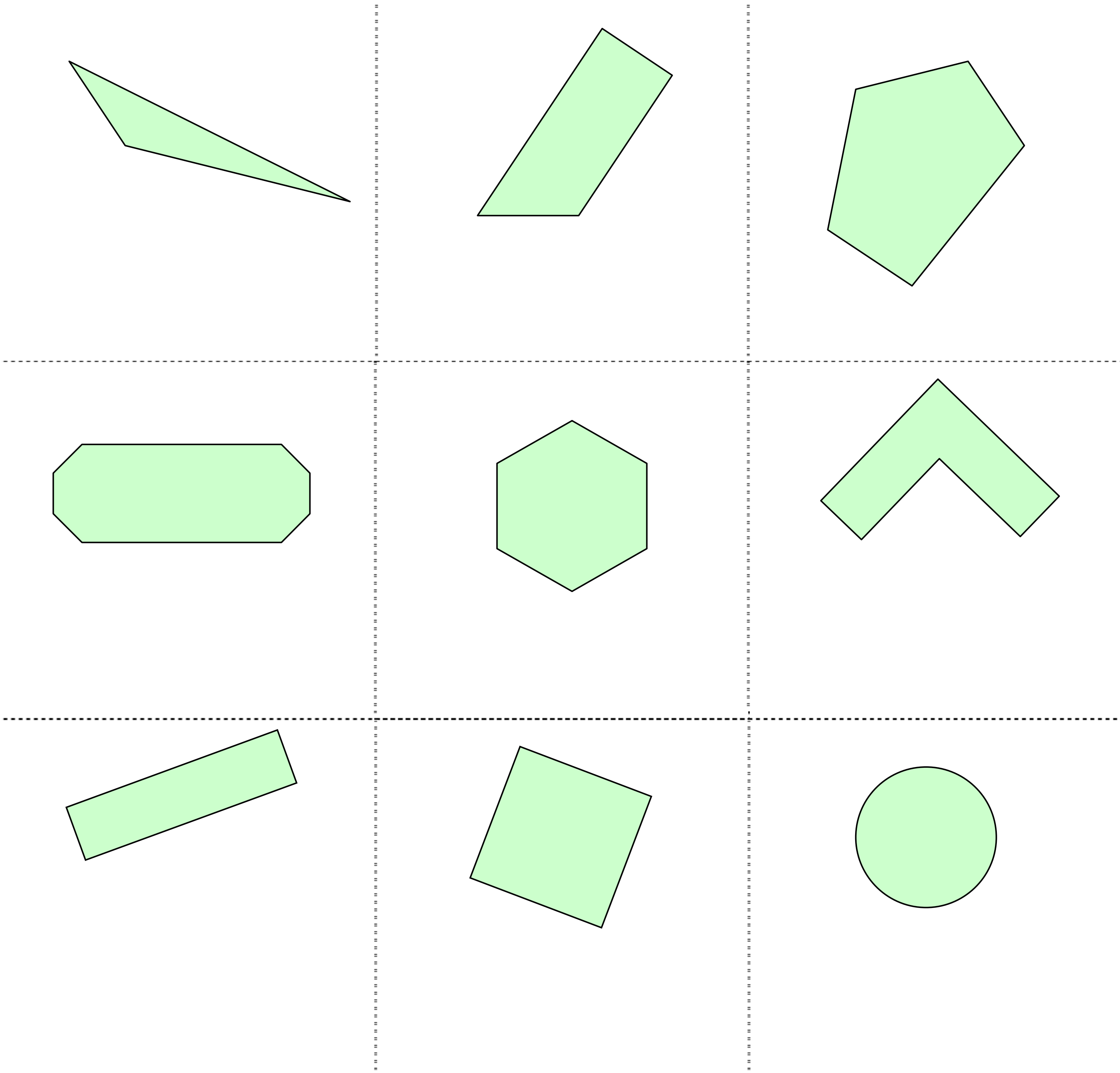
3. Before using pictures of 3-D shapes, place the set of 3-D shapes in a position where all the students can clearly see them.

- Ask the students to look at the shapes and think about how they are the same and how they are different.
- Choose a shape such as the cube and ask the students to say all the things they notice about it. In the discussion the students are likely to say, in some form, that it is square. Use this as an opportunity to introduce the idea of faces.
- Ask how many faces it has (six) and ask about the shape of each face. Take another shape such as a triangular prism to contrast. The name does not matter at this stage but unless students see a shape where the faces are not all the same, their understanding may be limited.
- Ask how many faces the prism has (five) and what shape each face has. Repeat by using the cylinder. This is important as it raises the issue of a curved surface rather than a "flat" face.
- Return to the cube. If the students use the word corners, use this as an opportunity to introduce the term vertices and ask how many vertices there are (eight). Go to the triangular prism and ask the same question (six), then the cylinder, which does not have any vertices.
- If the idea of edges has not already arisen in the discussion, focus attention on it as the third part of the cube and ask how many edges it has (twelve). Repeat for the prism (nine) and the cylinder (two). Ask how the edges in the cylinder are different (they are not straight). Again a range of examples assists understanding.

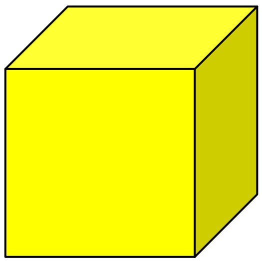
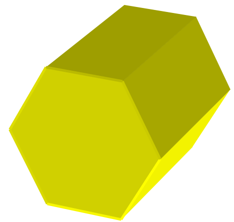
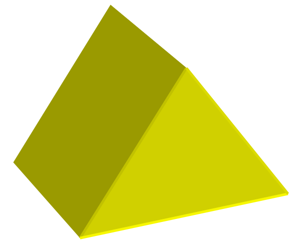
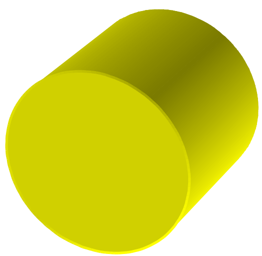
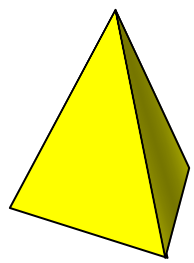
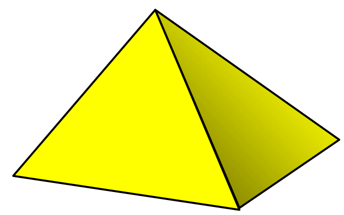
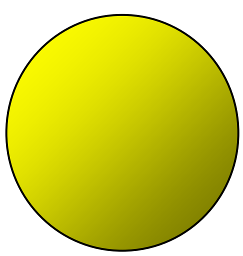
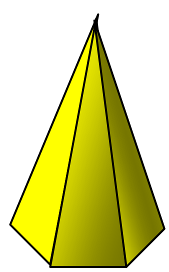
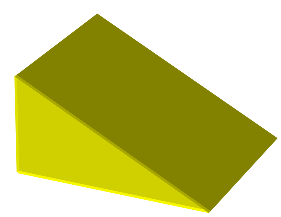
4. Use pictures on the cards that match the set of 3-D shapes and choose three students to come to the front for celebrity heads. The students ask their questions and may identify the shape by pointing to the 3-D object rather than naming it, though it is also all right if they do name it.

5. At the end of the activity, to consolidate the learning, ask the students what they have learned.

- Write the key words on the board.
- Make sure in the discussion the ideas of faces, edges and vertices are raised. Names of the shapes might also be used.
- Choose one of the 3-D objects and ask the students to tell you all they can about it.



blackline master

Prototypes

Note: This activity is designed for teachers.

Materials:
- A copy of blackline master (see p. 45).
- Overhead (p. 44).

Grade: Teachers: while not directly an activity for children, this focuses on describing and identifying 2-D regular and irregular polygons and images people have of shapes.

Activity Description: The purpose of this activity is to raise awareness of the issue of the use of prototypical shapes in the presentation of shape to children. While children, and indeed adults, may know a formal definition of a particular shape, their image of the shape may be limited to the prototype.

Background Information: Recognition of shape is often related to the visual image that a person associates with the word. An example of this problem was seen recently when a group of mathematics graduates at a university were asked to draw a hexagon. One student asked others at the table for an eraser. When asked why, he said that his hexagon was not exactly right. What he meant was that it was not a perfect regular hexagon. When shown an L-shape, many of the students did not identify it as a hexagon because it did not look like one. Awareness of this problem is important for teachers as they are the ones who orchestrate the presentation of the shapes to children. Further information is integrated with the activity description.

The Activity:

1. Ask everyone to draw a triangle and a hexagon.

2. When everyone has finished their drawings, ask participants the following:

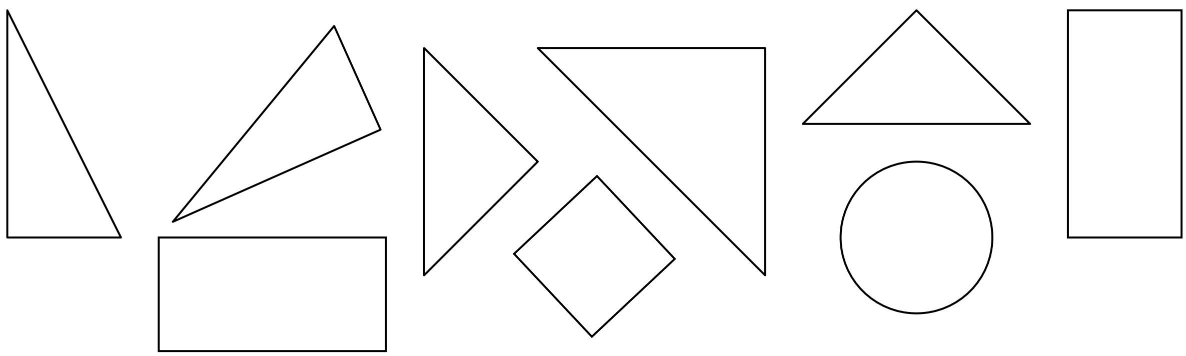
- *Raise your hand if you had a horizontal line in your triangle?*
- *Keep your hand up if the "other" vertex was below the horizontal line.*
- *Was there a vertical line in your triangle? Raise your hand if there was.*
- *Keep your hand up if there was a horizontal line as well.*
- *Was there a right angle (or nearly) in your triangle?*
- *Was your triangle close to equilateral?*
- *Was your triangle isosceles?*

3. Now look at your hexagon.

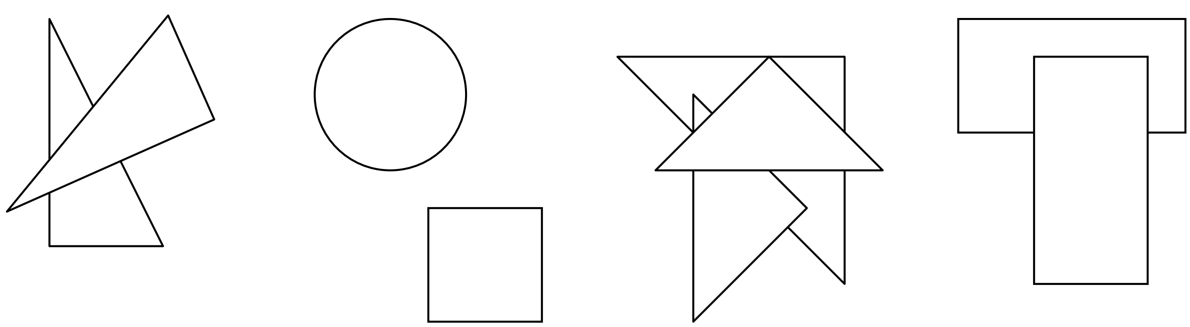
- *Was there a horizontal line in your hexagon?*
- *Was there a vertical line in your hexagon?*
- *Was your hexagon shaped like the beehive hexagon?*
- *Did anyone have a hexagon with a right angle in it? Two right angles? Three right angles? More than three right angles?*

4. Raise the issue of prototypes.

In the Early Numeracy Research Project (ENRP) Interview there is a question where the children are given the shapes below and asked to sort them into groups.



A number of children sorted the shapes into five groups as shown here.



When they name the shapes they said circle, square, rectangles, triangles and did not know what to call the first group. Even as high as Grade 4 there were some children who suggested that they could be called half triangles. They made statements such as "they are too long and pointy to be triangles" and "there is not enough space in here [pointing to the middle]."

5. Pass out the triangle sheet from the ENRP interview (see blackline master, p. 45). Ask participants to circle the numbers of the shapes that are triangles.

   - Once they have all completed the task, have them pair and share their answers with their partner, trying to seek agreement.
   - Then have them share their answers in table groups.

What might students say when trying to do this task?

Discuss why each shape is, or is not, a triangle.

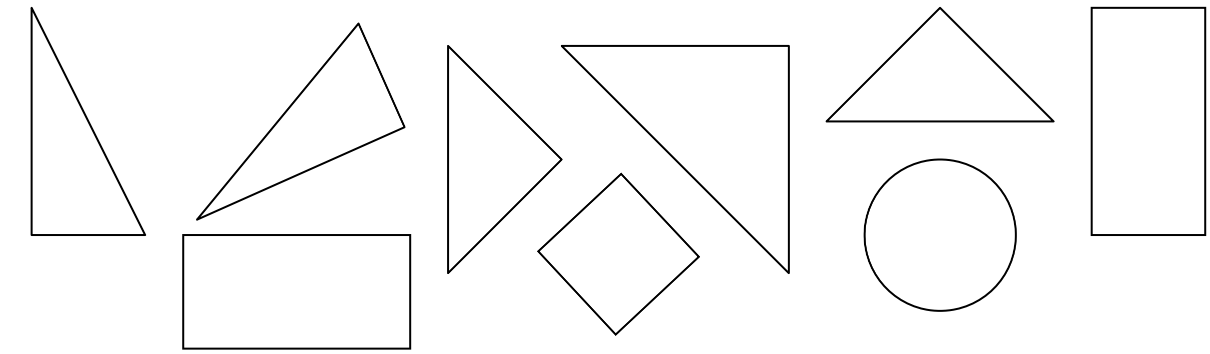
- Use the overhead projector slide to show shapes on screen while discussing.

Key points to make sure arise in discussion are:

- Many students have only a prototypical view of a triangle – “it is a triangle because it looks like one I have been shown.” The triangles shown to young students tend to be roughly equilateral or isosceles right angle triangles.
- There is a difference between the mathematical object triangle and a triangular shape.
- Often young students are given “triangles” with rounded corners so that they will not hurt themselves on the corners or are given the task of drawing around a triangle in which case their drawing has rounded corners, thus making them not triangles at all but triangular shapes.
- The musical instrument triangle is not actually a triangle but a triangular shape.
- To be a triangle, the shape must be planar and closed. The sides must be straight.
- To really learn a concept, one has to experience not only many different examples of the concept, but examples that do not belong. The experiences should push the boundaries of the concept.

overhead

A.



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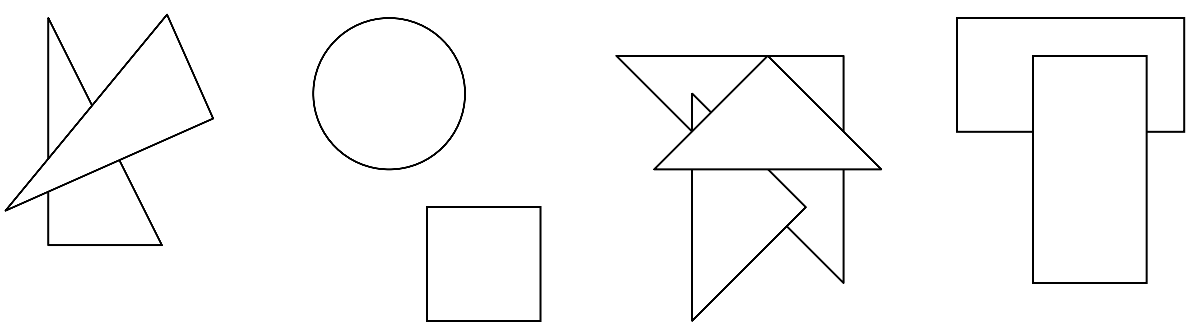
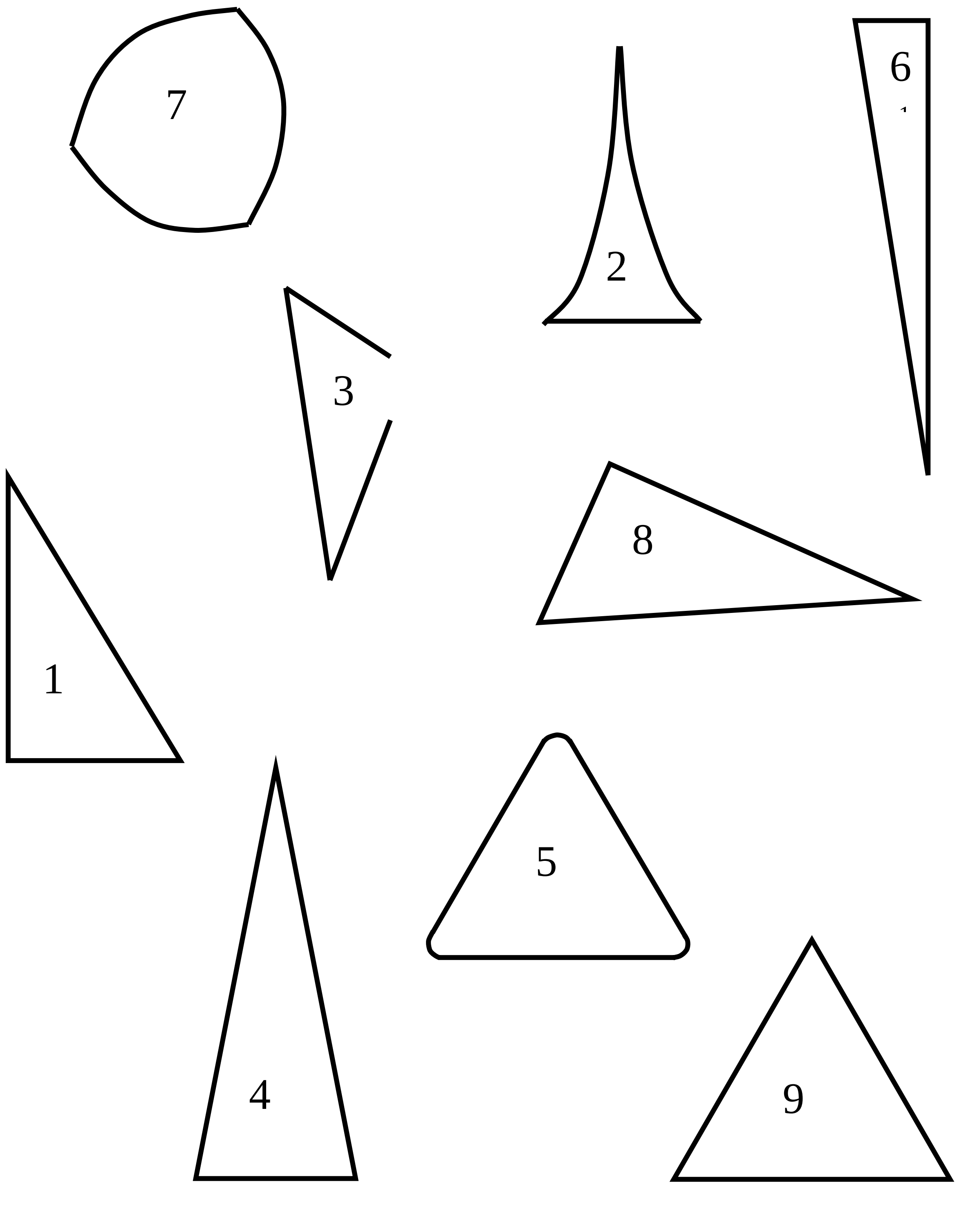



7
6
2
3
8
1
5
4
9

Making Shapes

Materials:
- Scissors
- Glue sticks
- Newspaper
- Nursery squares (kinder squares or origami squares) of two different colours.

Grade: 1, 2, 3

Learner Outcomes:

Grade 1, No. 3
Replicate composite 2-D shapes.
[CN, PS, V]

Grade 1, No. 2
Sort 2-D shapes using one attribute and explain the sorting rule.
[C, CN, R, V]

Grade 2, No. 6
Sort 2-D shapes using one attribute and explain the sorting rule.
[C, CN, R, V]

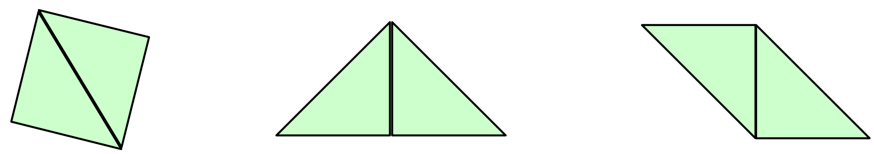
Grade 2, No. 8
Describe, compare and construct 2-D shapes.
[C, CN, R, V]

Grade 3, No. 7
Sort regular and irregular polygons, including triangles quadrilaterals, pentagons, hexagons and octagons according to the number of sides.
[C, CN, R, V]

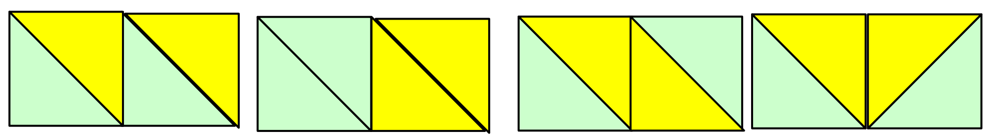
Activity Description: In this activity, students join triangles to make other shapes. The challenge is to explore the different shapes that can be made following the set rules. Having each make a shape and glue it onto newspaper, students then sort the shapes and discuss what is similar and what is different about them, trying also to name all the shapes made.

Background Information: The two triangles in the activity will make three shapes according to the rules: a square, a triangle and a quadrilateral (shown below). Although it is beyond the syllabus at this stage, students love learning about big words, particularly if they are fun to say. For this reason you might decide to

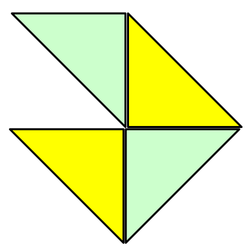
introduce the word parallelogram. I have heard young students saying “para llelo llelo llelo gram,” having fun with their tongue on the “llello.” It is also important as the use of the word parallelogram differentiates the shape from the square while the square is also a quadrilateral.

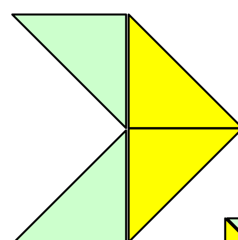


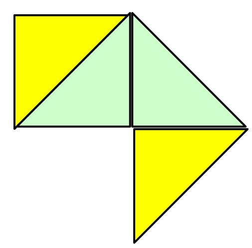
When students move to the second task with more triangles, there are many more options, some of which are the same shape but coloured differently, as in the first example, and some of which are different shapes.

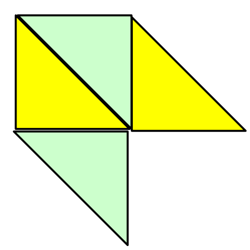


Here are four different hexagons

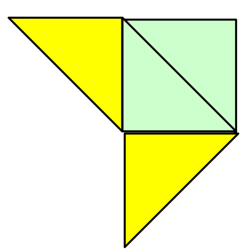








and one pentagon.



The Activity:

1. Demonstrate folding a square diagonally to divide it in half and cut it into the two triangles. Ask students to take a square of one colour and do the same.

2. Ask students to put their two triangles down on the table to make a shape following the rule. Explain that the rule is that the triangles must join completely along an edge from one corner to the other. You will need to model this so that the students understand both what is allowed and what is not allowed.

3. Ask the students to look around their table and to see how many different shapes were made.
   - Ask students (name one—Gillian maybe) to show their shapes on the overhead projector with your two triangles and to name the shape. For each, ask for a show of hands for those who had the same shape. Keep asking for other shapes and then ask, "how do you know you have found all the possible shapes we can make with two triangles using our rule?" The possible shapes (triangle, square and parallelogram) are shown above in the background section. Again, while parallelogram is not part of the syllabus, the word quadrilateral also includes the square so differentiation between the two shapes is needed.

Often with a group there will be four shapes named and shown:

- triangle
- parallelogram, though sometimes there is discussion about whether it is a rhombus. This raises the question of properties—what is special about a rhombus? Is it true here?
- square
- diamond—this should generate some discussion if it arises. Diamond is not a mathematical shape in the same sense as the other shapes mentioned here because diamond is defined by its orientation rather than just its shape.

  There may also be confusion about what a diamond is. Some people (and some dictionaries) say a diamond cannot be a square and so really define it as a rhombus that is not a square. Others do not. If we use that definition, this is not a diamond, but this also causes contradictions with things like the baseball diamond that is used in common speech.

It is better not to use the word diamond in this context at all but to use this opportunity to talk about shapes being the same shape no matter what their orientation. This is different to letters when reading, as there, if a shape such as a “b” is reflected, it becomes a “d” or a “p” and has a very different effect on a word.

If participants do not raise the issue of diamond, raise it so that all are aware of the problem. With students it is important to have the discussion about diamond and to teach them the mathematical terms. Often there are common language uses of words which are not as exact as the same words used in a mathematical sense. For example, in common speech it is not unusual for people to refer to a “bigger half.”

4. Make the triangle on the overhead projector and ask each student in the room to make that shape in front of them. Repeat for each of the other shapes.

5. Explain to the students that now that they know the rule we are using to join the triangles, they are going to try it with four triangles. Each student is to take a square of the other colour and make two more triangles, then using all four triangles, coloured sides up, make a shape. When they are happy with the shape they have made, they glue it onto a sheet of newspaper.

Take the students with their sheets to an open space around which they can stand and ask them to put their shapes down on the floor in groups (or fix to a large wall space). Ask them to explain the grouping rules that they used. This will show many different hexagons and pentagons with no regular ones. Ask them to sort with a different rule. Although hexagons and pentagons are beyond the Grade 3 syllabus, allowing the students to sort and to use the ideas of same and different with a range of shapes provides a rich experience. The definitions of pentagon and hexagon and the names are not necessary at this stage.

Note: Often in discussion with students, many ideas will arise that are beyond the outcomes expected at that level, but they should be explored when they arise. Students need to experience and play with ideas when they arise naturally quite a time ahead of when they are expected to achieve the formal outcome.

Points which might arise in discussion include:

- symmetry; this could be both symmetry of shape and symmetry of colour
- for the teachers rather than for the children, there is reflectional symmetry and rotational symmetry; the parallelograms, for example, demonstrate rotational symmetry but not reflectional symmetry
- some of the shapes will be concave. The use of this word is not important for students at this level but teachers should be aware of differences
- what is necessary to define a shape

- names of many shapes, e.g., an isosceles trapezium (trapezoid).

Ask participants to suggest further investigations. Possibilities include:

- *"How many different shapes are possible and how will we know if we have found them all?"*
- *"How many different shapes and colours are possible? Taking the aspect of colour into account expands the possibilities."*

6. It is really important with children (and adults) to consolidate the learning by asking them to reflect on what they have learned and, in a short, focused discussion at the end of the activity, draw out the main points.

One key issue that should arise in this activity is the idea of regular and irregular shapes. The square is a fully regular shape in that both each side and each angle are exactly the same. The triangle they make with the pieces here is irregular as not all sides are the same length. Also not all angles are the same. While the triangle has symmetry, the fact that some of the sides and angles are different makes it irregular. The hexagons that arise in this activity are all irregular. Too often children (and adults) connect the image of a regular shape with the name of the shape and do not see an irregular hexagon, for example, as a hexagon. Discussion with the teachers about a regular shape having all of its sides the same length and all of its angles the same is important. This also carries over to 3-D where there are only five completely regular polyhedra with all of their faces, vertices and edges identical. Though not a polyhedron, a sphere is also completely regular.

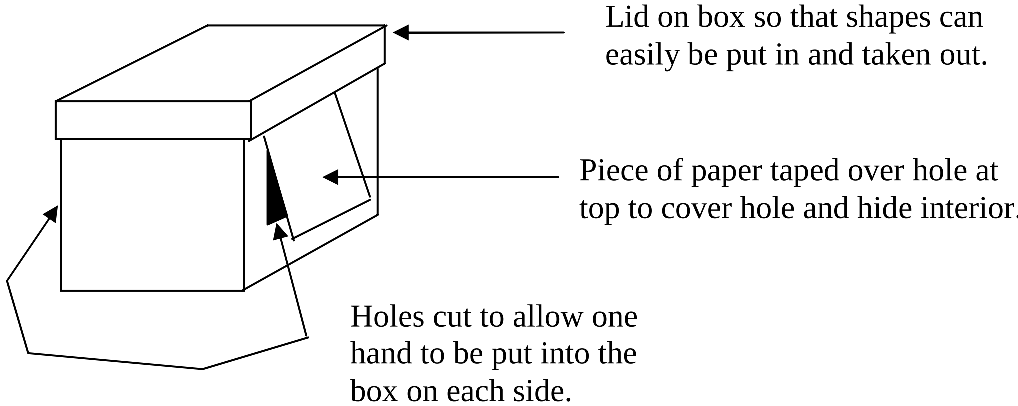
Extension:

If the rule is changed so that they must join along an edge and touch at a corner, but do not need to have both corners meeting, how many shapes can then be found?

Feely Box (or Black Box) Geometry

Materials:

- Overhead projector
- Paper and pencil for drawing
- Overhead (see p. 59)
- A cardboard box about 25 × 35 × 35 cm with a lid with holes and flaps as in the diagram for each group. The box can easily be made from a copy paper box. A hole is cut in each end to enable the students to put their hands in the box – one from each side. A flap of paper is taped so that it is fixed only at the top but covers the hole and restricts seeing the contents of the box.



A collection of shapes as described below each variation of the activity. Attribute blocks and simple 3-D models can provide a start. Other 2-D shapes can be cut from heavy cardboard. It is good to have a number of different samples of each shape.

Shapes

Use a range of 2-D shapes including different types of triangles and quadrilaterals, and nonprototypical ones. Choose the shapes to suit the grade with which you are working. When the class is really using properties (such as the number of sides, the type of corners and whether sides are equal in length), to identify shapes, you should extend the shapes to include unusual ones like the ones below. The properties listed for each shape include some in italics which are not part of the syllabus at this level. The focus should be on the properties which are not in italics, but the others are included here for information in case they arise.

Grade: 2, 3

Learner Outcomes:

Grade 2, No. 7
Describe and compare 3-D objects including cubes, spheres, cones, cylinders, pyramids.
[C, CN, R, V]

Grade 2, No. 8
Describe and compare 2-D shapes including triangles, squares, rectangles, circles.
[C, CN, R, V]

Grade 3, No 6
Describe 3-D objects according to the shape of their faces, and the number of edges and vertices.
[C, CN, R, V]

Grade 3, No. 7
Sort regular and irregular polygons according to the number of sides.
[C, CN, R, V]

Activity Description:

This activity is specifically focused on visualization and the properties of shapes. It uses a box with flaps to hide the contents but allow hands so that the students cannot make use of their eyes. Children who are blind develop their own forms of visualization and often carry strong mental "pictures" because they cannot rely on seeing with their eyes. The feely box gives students experiences of using touch to see geometric objects. The task also requires the use of language to connect the visual picture with properties of shapes.

Identifying shape by using the sense of touch rather than sight can assist students to visualize the shape through its properties.

Background Information:

While not specifically connected to an objective at the Kindergarten level, this activity encourages exploration of shape and visualization at all levels. There are many adaptations of the task possible so the task can actually be used effectively in all classes from Kindergarten to Grade 9.

Van Hiele's levels of students' learning in the area of geometry:
Level 1. Recognition: "The student identifies, names, compares and operates on geometric figures according to their appearance."[1]

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1. Fuys, D., Geddes, D. & Tischler, R. (1988). The Van Hiele model of thinking in geometry among adolescents. (JRME Monograph Number 3) Virginia: NCTM.

Level 2. Analysis: "The student analyses figures in terms of their components and relationships among components and discovers properties/rules of a class of shapes empirically. The figures are identified in terms of their properties rather than just their appearance."[1]

Level 3. Ordering: "The student logically interrelates previously discovered properties/rules by giving or following informal arguments."[1] This means that as well as the figures carrying properties, the students are interrelating these so that, for example, a triangle having two equal sides implies that there are two equal angles.

Level 4. Deduction: "The student proves theorems deductively and establishes interrelationships among networks of theorems."[1]

Level 5. Rigour: "The student establishes theorems in different postulational systems and analyses/compares these systems."[1]

Activities:

Activity 1

Students love the mystery of discovery of the unknown. This is a twenty-question type of activity that encourages the use of geometric language and requires visualization. The first time doing it should be with the whole class.

1. One student reaches into the box with one hand on each side and picks up a shape. The student with hands in box is restricted to answering only *yes* or *no* to questions from the group trying to draw the object. This forces the students to think more about properties that enable identification, thus moving away from the prototypical image towards van Hiele's Level 2, the use of properties in identifying shapes.

   - Sometimes in this situation the student answering may not understand the question so it is necessary to allow him or her to answer, "*I don't understand your question. Please ask the question again in another way*" if he or she does not understand the question.

   - Sometimes the student might understand the question but not know how to find the answer, in which case the response would be, "*Please tell me how to find that out.*" This then puts it back to the questioners to think about both their language use and their visualization of the shape. (See overhead p. 59.)

- It is a good idea to write these responses up on the blackboard or whiteboard or on an overhead so that the students can clearly see them.

2. It is important, after a few questions, that the class stops and discusses the sorts of questions that they could ask and what questions provide useful knowledge that enables them to determine the nature of the shape and thus draw it.

For example:

- Are all the sides straight?
- Does it have exactly three sides?
- Does it have more that four sides?
- Are any of the sides the same length?
- Does it have square corners (right angles)?

Sometimes language can be misleading – for example, if a shape has six sides, a student who answers "yes" to the question "Does it have four sides?" is actually correct but "no" would be the expected answer. This is not an issue that needs to be raised with students unless you notice a misunderstanding.

Questions such as "Does it look like …?" are not as useful usually in determining the shape. You might need to ban the use of the eyes so that a student cannot hold up a picture or draw in the air and say "Does it look like this?"

It depends on the class and your specific aims as to what language you allow. For example, young students might ask "is it a …" and name the shape, such as square, triangle. The task can easily be altered for different purposes by restricting the types of language used. Banning the use of the word *like* or *similar to* or such equivalents can overcome the use of van Hiele's Level 1 ideas of "it is like the door."

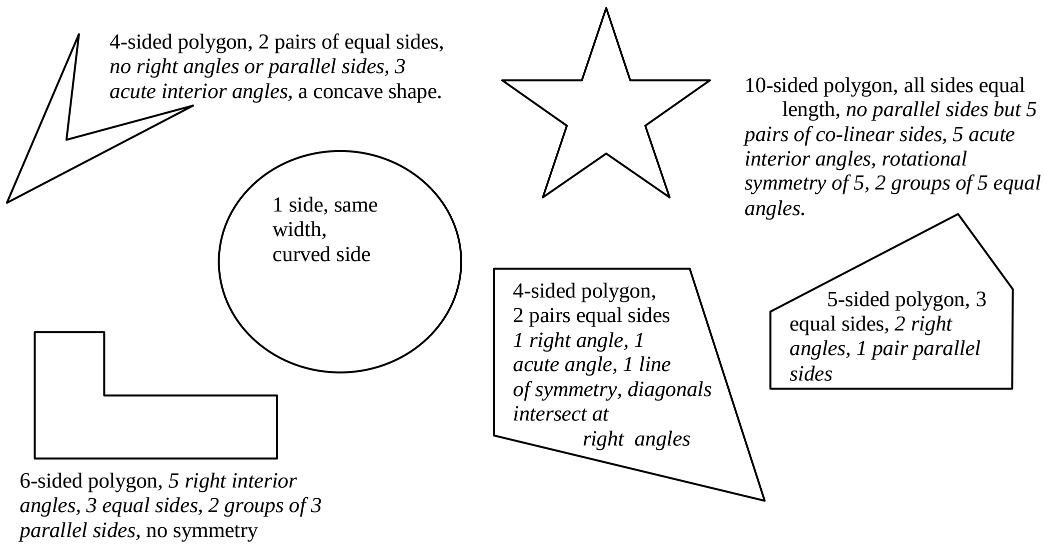
It is really important to be clear in the focus of the task and to recognize that the aim is to move the students towards using properties to identify and classify shapes. Sometimes the focus may be on common language and specific geometric names are banned, e.g., the use of "Is it a right-angled triangle?" is banned, but descriptions of properties are allowed, e.g., "Does it have three straight sides?" and "Does it have one right angle?" On other occasions the use of geometric terms and language are encouraged to show the value of specific, well-defined terminology.

3. In a classroom using a number of boxes, once the students understand the process as it was modelled in the whole class, groups of four to six can be formed at tables and the activity take place in table groups. This allows all students to experience the different roles. When they have emptied the box of shapes (it is good to have enough shapes in the box for at least two each), ask them to sort the shapes into groups according to some rule and explain their rule to you. They might sort into: symmetrical and nonsymmetrical; regular and irregular; triangles, quadrilaterals and others; or shapes with a curved side and shapes with all sides straight. In discussion stress the aspects of "What is the same about the shapes?" and "What is different?"

4. Consolidation of the concepts is important and the class needs to be drawn together for focused discussion reflecting on their learning. Discussion during and following this activity can focus on language.
   - What types of descriptions were most helpful in enabling the students to see the shape in their minds? For children, the question asked could be, *"What helped you to see what the shape was?" or "What helped you picture the shape?"*
   - Were there any problems with interpretation where the questioner and the person holding the shape had a different understanding of the words used? For children, you might ask, *"What difficulties did you have?"*
   - What specific geometric language was used in comparison to the common language used? With the class, it is a good idea to ask about the language they used and to write geometric terms such as sides, angles, triangle, quadrilateral, square, on the board.
   - The student with hands in the box is translating from the actual object to language and the other students are translating from the language to a representation of the object.

Everyone involved in the activity is using visualization to make the connections between the language and what is not seen.

Shapes
Use a range of 2-D shapes including different types of triangles and quadrilaterals, and nonprototypical ones. Choose the shapes to suit the grade with which you are working. When the class is really using properties (e.g., the number of sides, the type of corners and whether sides are equal in length, to identify shapes, extend the shapes to include unusual ones like the ones below. The properties listed for each shape include some in italics which are not part of the syllabus at this level. The focus should be on the properties which are not in italics, but the others are included here for information in case they arise.



The students love the puzzle aspect of the feely box.

Activity 2

1. With very young students, put a large collection of triangles and quadrilaterals into the box, making sure there are many different shapes among them. Sitting around in a circle, pass the box from student to student and ask each student to try to take out a triangle (make sure there are plenty). This gives you a clear idea of students who are having difficulty with the three sides, three angles (points) connection to triangles. You will also notice those who do not visualize but recognize when they see them.

2. Once each student has a triangle, then the students can sort the triangles. Just the experience of seeing many different-shaped triangles is valuable. There are many different ways of comparing the triangle. For example, they can discuss size and connect that to developing concepts of length and area.

3. Consolidate the learning by asking them to tell you what is the same about all the triangles. Ask them also how they differ. Use this to focus attention on the properties that make them triangles.

Activity 3

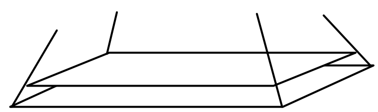
1. Place a set of geometric solids where all the students can easily see them. Put another set into the box. The student holding the object inside the box answers the questions as before and the students try to identify which of the objects it is. This can be used to encourage the use of geometric language such as faces and edges.

In the early stages, and with young students, it might be that the person holding the object just tries to identify it as, for example, *the third one* or *the one on the left*.

Older students might be able to use the names of 3-D shapes.

2. Students love using the box also just to identify common objects from the classroom or from home. Place a number of ordinary objects with geometric-like shapes in the box – e.g., a can of food, a small box such as a packet of foodstuff, a Toblerone box, a blackboard eraser.

Be careful. The plastic solids can cause some problems as often the base has a rim and the hole in will be confusing if the describer wants to be very accurate. I saw one student in a class claim a square pyramid had 9 vertices because of the rim around the base. The student was correct in the description – the shape was not accurately a square pyramid, but it did cause a lot of confusion.



Possible Variations

A. You may want to start by having one student reach into the feely box and hold one of the solids/shapes without taking it out. The student then describes the features of the solid without saying its name. The other students try to guess which solid the student has selected.

You may want to display the solids on the teacher's desk so students can refer to it as they try to guess the solid/shape the student has selected.

B. This technique can be varied by having one student select a solid/shape from the teacher's desk and a second student must find that shape/solid in the feely box by feeling. When they think they have the matching shape/solid, they take it out of the box.

C. Another version has the students match the solid to a picture of the solid. Place a set of 2-D shapes or 3-D solids in the feely box. Place a set of picture cards showing the same shapes/solids face down on the table. Have the students turn over a card and feel in the box to find the shape.

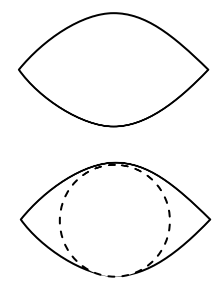
All of these tasks with the feely box can be varied to suit the particular group of students and the particular objectives of the course. **The discussion and the use of language are a very important part of all these tasks.** The feely box provides one way of aiding visualization and helping to consider properties along with the shape, thus moving to van Hiele Level 2.

Extension

1. For some three-dimensional shapes, the task for the group may be to work in small groups to build the shape so that the representation is another object with the same properties (and same name) but the material and size may be different.

For example, place a triangular prism or a cube in the box. Using pieces such as geoshapes, ask the students to build the shape they have identified.

I am indebted for this idea to Geoff Giles who at ICME in 1984 put simple geometric puzzles in such black boxes, giving the possibility of another type of task that is more individual problem solving. For example, try passing a regular tetrahedron through a hole like this when you cannot see.



The size should be carefully chosen – the diagonal of the cross section square in the centre of the tetrahedron must be slightly less than the diameter of the dotted circle.

Yes

No

I don't understand your question. Please ask the question again in another way.

I don't know. Please tell me how to find that out.

Create a Copy

Materials:
- Pattern blocks
- Overhead projector
- 3-D shapes and blocks for building.

Grade: 1, 2, 3

Learner Outcomes:

Grade 1, No. 3
Replicate composite 2-D shapes and 3-D objects.
[CN, PS, V]

Grade 2, No. 3
Describe, compare and construct 2-D shapes.
[C, CN, R, V]

Grade 3, No. 6, 7
Describe 3-D objects according to the shape of the faces and the number of edges and vertices.
[C, CN, PS, R, V]

Sort regular and irregular polygons according to the number of sides.
[C, CN, R,V]

Activity Description:

The purpose of this task is for students to copy simple composite shapes created by another student, then to copy a shape created by the teacher, identifying its component parts. It allows them to predict and select the 2-D shapes used to produce a composite 2-D shape. There is also opportunity to describe 2-D shapes and 3-D objects.

Background Information:

One interesting aspect of this task to watch for is the aspect of orientation. In creating a copy, does the student make the shape the way they see it from where they are sitting or do they make the shape as it is oriented to the other student? This is the reason for having the students sit side-by-side. If students are sitting around a table, some will orient the shape as if it were just a translation while others will orient the shape as if they were seeing through the other student's eyes.

Asymmetric shapes are more difficult for students to replicate and sometimes the students will form mirror images.

The Activity:

Activity 1

1. Two students sit side by side. One student chooses two pattern blocks and arranges them together. The other student then creates a copy.

2. After doing this a few times, change the activity to be three pattern blocks.

3. One student at the table makes an arrangement with a hexagon, a triangle and a rhombus and everyone else copies that one.

   - Look at the copies – do they have the same orientation as the original? Or, do they have the same orientation *relative to the student who made them*? In other words, when making the shapes, are the students doing it as if they are seeing through the maker's eyes or just translating it?

   - This raises the issues of perspectives and differing frames of reference. Later in these students experiences, they will need to be able to visualize or see things from different perspectives and this gives the teacher a window on how young children do this.

4. The task can be made more complex as students are ready for it. This can be done by slowly increasing the number of shapes used in the shape to be replicated. Ask the students to try it with four blocks. When they can do it with four, increase it to five blocks.

   In later years the challenge may be to make a mirror image or a quarter turn rotation. Ask the teachers to try this in the workshop. While not specifically related to an outcome at grades K–3, this is a worthwhile exploration of transformations that prepare the teachers well for seeing where the activity might go in future years

Extension – Using geometric language

An alternative activity is to have one student sitting behind a screen (or with back to his or her partner). This student builds a shape using pattern blocks and describes it to his or her partner, who then tries to duplicate the shape.

Activity 2

1. Make a shape with three pattern blocks on the overhead projector but do not turn on the light. Use opaque pattern blocks so that colour does not take over as the dominant indicator but the students' attention must focus on the shape.

2. Explain to the students that they are going to try to make the same shape as the one you show on the overhead projector, but that they are going to have to decide which shapes to use. You are also not going to show the picture to them for very long so they are going to need to try to keep a picture of it in their head while they try to make it. They are to look at the picture but not to start making it until you turn the projector off.

3. Turn the overhead projector on for about 15 seconds the first time (count slowly to 15). Tell the students to try to make your shape. This task asks the students to decide what shapes to use but also requires visual memory.
   - When most of the students seem to have finished it, turn the projector on again and ask them to check their shape. Ask them to think about how the shape they made is the same and how it is different to the one you made.
4. Repeat the task. Ask students what they paid attention to in order to help them replicate the shape. For children, ask what they noticed about the shape.

Activity 3

1. This is like Activity 1 but uses 3-D blocks instead of pattern blocks. Two students sit side by side. One builds a shape with three blocks. The other student then replicates the same shape. They then change roles.

2. Extend the students by telling them they can use four blocks in building their shape and not all of the pieces they use can be the same—i.e., at least one block must be a different shape. The activity can then be extended to five blocks.

Extension:

Students work in pairs. Have a student create a composite 2-D shape or 3-D object behind a cardboard screen. The student must then describe the shape/object to their partner and the partner then recreates the shape/object. This really focuses on language. Ask students what kinds of words or phrases made it easy to follow and what where some of the challenges. The emphasis is on using precise language and giving clear descriptions. While consolidating the language in this activity it is a good idea to write key words and phrases up on the board for later reference.

It is important to begin by using only three to four shapes/objects. As students get more proficient at this, you may add more shapes/objects.

Extension to drawing:

Drawing shapes is one way students show what images they have, but they are just learning to draw and the drawings do not always match the images we expect to see. The same is true for adults. I have a very clear image of a horse in my mind but I have a lot of trouble drawing that horse on paper. The last task, which encourages both visualization and the deconstruction and construction of a 2-D composite shape, can be extended to a drawing task using Quick Draw.

Quick Draw

Materials:
- Overhead of geometric shapes for students to draw (see pp. 64, 65)
- Blank paper for covering

Grade: K–6

Activity Description: The purpose of this exploration is to help students develop the special sense and ability to form mental images.

Background Information: Please see *Imagery and Mathematics Learning* in the Research section.

The Activity:

1. Tell the students you will show them a shape on the overhead for only a few seconds. Ask students, *"Make a mental picture of the shape so you can draw it after I turn off the projector. Read?* On the count of three, one, two, three."

2. Turn on the overhead projector. Show the shape for three seconds and then turn the projector off. Have students draw the shape they saw.

3. When most students have drawn all they can, tell the students, *"I will now give you another look at the shape so you can revise your drawing if you want to."*

4. Turn on the overhead again and let students look at the shape for another three seconds. Turn off the overhead.

5. Allow students to revise their drawing.

6. The discussion of their drawings is the heat of the activity. Ask students, *"What did you see and how did you draw it?*"

   - Encourage students to talk about their drawing and what they saw when they looked at the shape. They should also share their strategy for reproducing the shape.

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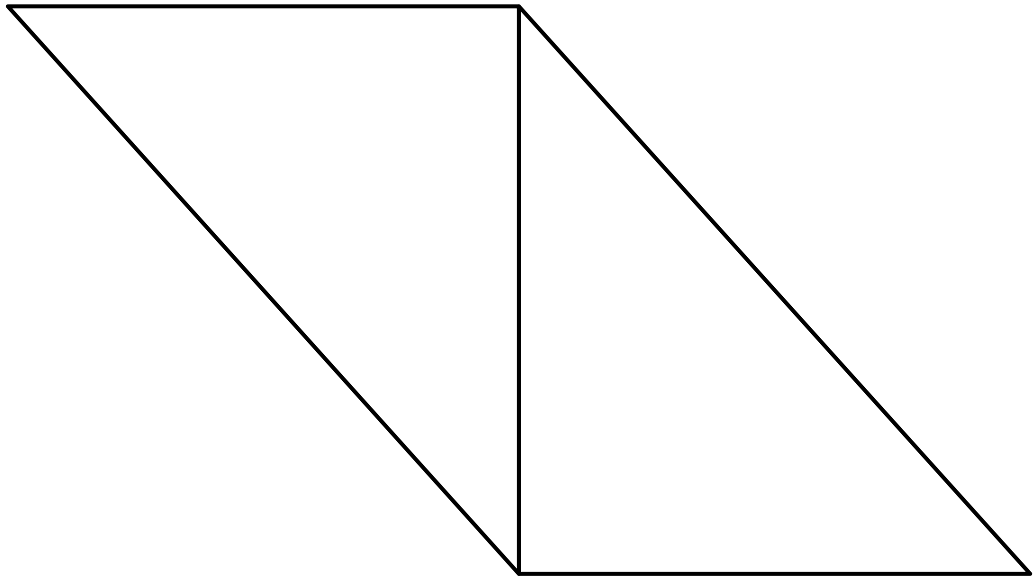
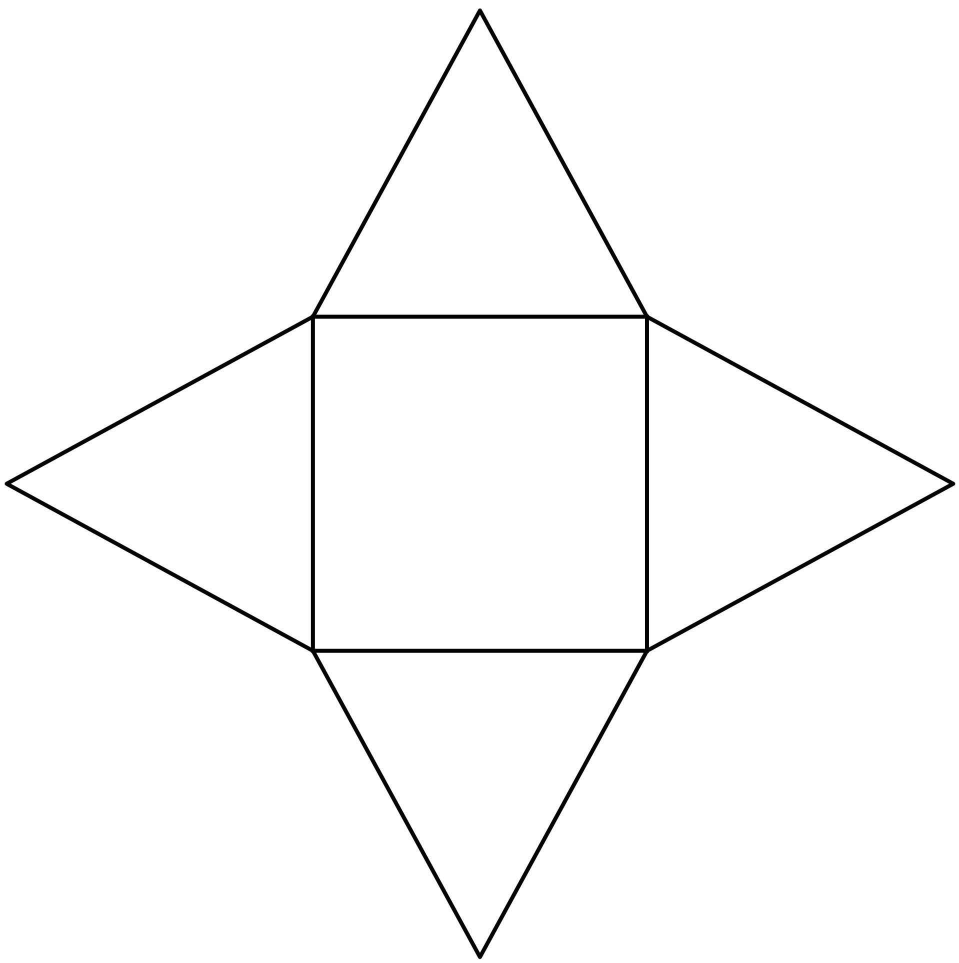
overhead





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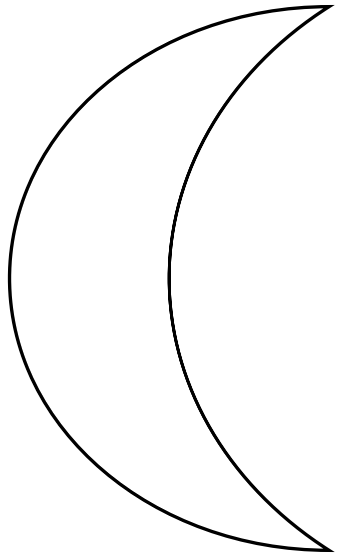

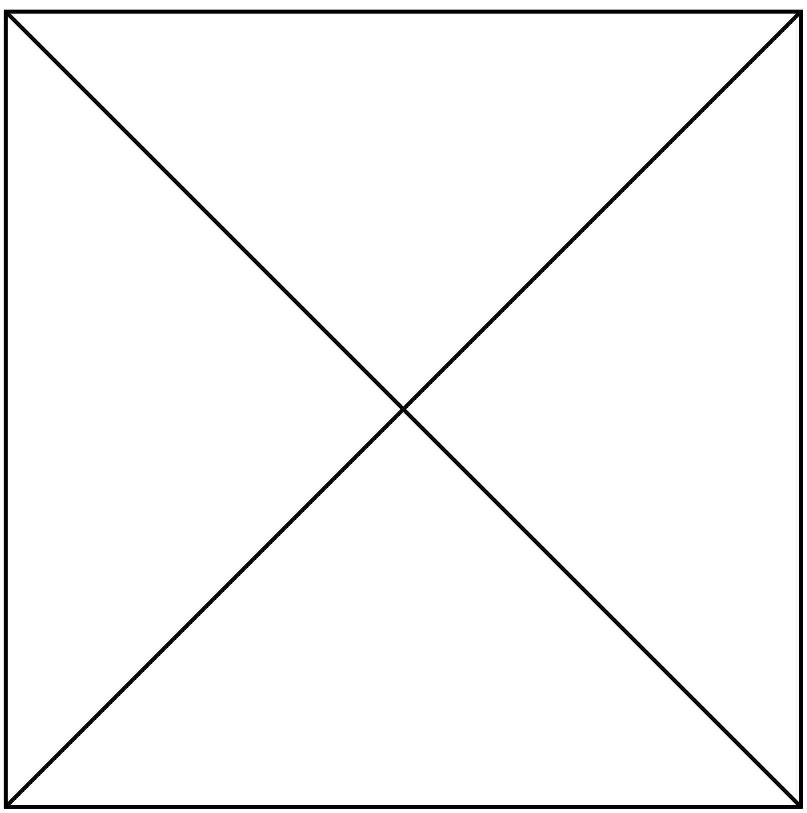

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Using Technology Tools

Materials:

- Computer with either Cabri Geometry or Geometer Sketchpad connected either to a data show or a smart board. The smart board works really well as, instead of using a mouse, the students physically use their hands to drag vertices and shapes.

Grade: 2, 3

Learner Outcomes:

Grade 2, No. 8
Describe, compare and construct 2-D shapes.
[C, CN, R, V]

Grade 3, No.7
Sort regular and irregular polygons according to the number of sides.
[C,CN, R, V]

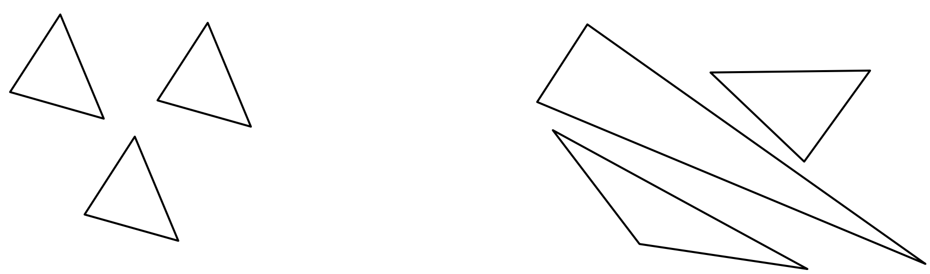
Activity Description: The purpose of this exploration is to encourage students to reason about what is the same and what is different about shapes.

Background Information: These computer programs have been used everywhere from pre-school children to university students. The programs enable the drawing and construction of shapes. Once a shape is constructed, depending on the process of construction, using the mouse, a corner of the shape can be picked up and the shape altered. If a smart board is being used, the students can actually make the changes with their hands. This is very powerful, as it enables them to see shape dynamically and identify the invariants.

The Activity:

1. Create three triangles on the screen.

2. Challenge the students to alter the shapes by picking up one corner. Each student can be challenged to make them different. Each time the questions are *"What shape are they?"* and *"How do we know?"* What is really being asked is "What is invariant?" and "What is it that makes the shape a triangle?"
   - The diagram below shows three triangles on the left and how they might be altered by the students in the diagram on the right.



- In the discussion about what is the same and what is different about the shapes, the invariant properties are that there are always three sides and always three corners or angles (hence the name tri-angle). They are different in that they are “narrower,” “longer,” “more pointy.” “bigger,” “have more space in the middle,” etc. The students’ language will be descriptive rather than formally geometric.
- The strength of the use of a computer program like this is that the students see dynamically nearly all possible triangles in the course of the activity.

3. Repeat the task with a quadrilateral, then later a pentagon, a hexagon, etc. Each time question the students with *“How do we know?”* Have them identify the invariant properties, which for quadrilaterals is the four sides and the four corners or angles.

4. Consolidate at the end of the activity by asking the students what are the key things that make a shape either a triangle or a quadrilateral. They should mention the properties that they found invariant.

Extension for the teachers

- Create four different triangles – an equilateral triangle, an isosceles triangle, a right-angled triangle and an ordinary triangle (which may be scalene).
- Challenge one of the participants to come and move corners of these and to see what they can discover about them. The shapes need to be carefully constructed so that their properties are stable in the dynamic environment.
- They should be able to identify that the equilateral triangle is always an equilateral triangle though its size may change.
- Similarly, for each of the other triangles, ask how they know that each triangle is the type that they identify by asking what invariant properties each has.
- With the workshop participants, some of the special quadrilaterals might be investigated.

Exploring Shape and Objects

Materials:

- A large collection of common objects from around the home or classroom, divided up into containers so that each could be used by a small group of four or five students. This might include buttons, containers from shops such as cans and small boxes of a variety of shapes including a matchbox, cardboard tubes from paper rolls, blocks (both wooden and plastic), dice and other materials from the mathematics cupboard

Grade: K

Learner Outcomes:

Kindergarten, No. 2
Sort 3-D objects using a single attribute.
[C, CN, PS, R, V]

Kindergarten, No.3
Build and describe 3-D objects.
[C, CN, PS, R, V]

Activity Description:

This activity specifically focuses on sorting objects and using different attributes and properties to enable the sorting. In the discussion, attention is drawn to shape. The final part of the activity is to encourage description of a composition of a small number of objects which may be built into a structure or just arranged in some way.

Background Information:

The objects that might be used have both attributes, which includes such things as colour, size, thickness, texture, and the material from which it is made and geometric properties such as square faces or curved edges. There are many different ways of sorting the collection of objects into groups and young children will do it in many different ways. The richness of this task is in the discussion and their sharing of the rules or classification system they used for sorting.

The Activity:

1. With the class sitting in a large circle on the floor, tip one container of objects out in the centre of the group. Ask the students to look carefully at the objects but not to touch yet. They are to look for objects that are the same in some way. Go around the group and ask each student to pick up two objects that are the same in some way and tell the group how they are the same. During this session, stress the types of attributes that are being mentioned. They

might include colour, size, material (wooden), use (buttons, food), number (two holes in buttons), or shape, depending on what students notice and what is in the collection.

2. With the students sitting in small groups around a collection of objects, ask them to each take a group of objects from the collection that have something the same about their shape. Give each student a sheet of paper and ask them to draw what it is that is the same and arrange their collection on the sheet. Some of the students might like to count the objects in their collection.

3. Call the students' attention again and in the circle have them display their collections. Go around and ask each student to explain to the group in what ways their shapes are the same. In this way consolidate the learning by focusing on shape language and ideas raised. For example, some children might talk about circles or curved parts. Others might raise words for other shapes such as triangles, squares or rectangles. Some might just say they are all the same shape or they are all boxes.

   - Sometimes if a particular shape is mentioned, ask the students to find another object that could be added to that collection. For example, if a student says they are all round, ask the students to find another object that could be added to this collection of round things.
   - This activity gives a lot of opportunity to use language. If the student says they are all round, then ask if anyone knows the name we use for a round shape and show them a circle (which may be on one of the larger objects so all can see it), drawing around it with your finger. If the students do not use the name, it is an opportunity for you to tell them the word, but usually someone knows *circle*. In this way language can have attention drawn to it and sometimes new language can be introduced. Do not introduce the language if no student has suggested the idea through their collections. Rather, take note of what shape ideas the students did not use and plan a collection of material for another occasion that is more likely to draw attention to that concept.

4. Ask the students to choose three objects from the collection and have them arrange them in some way. In the group, choose a student to describe one of the arrangements. The other students try to identify it.

   - For example, they might all be asked to choose three blocks to build a tower. Then in the group, the students take turns to describe one of the towers. When they have finished their description, the others try to identify which one is being described. If they are unsure, then encourage the students to ask a question about the tower to help them to identify it.

Extension:

There are many ways these collections can be used for sorting. It is good to just ask the children to sort them and then ask them what is the same about each group.

Shapes Around Us

Materials:
- To do this in a class, the teacher needs access to a digital camera and the children need a notebook for each group to record their findings.
- Overhead projector and overhead (see p. 73).

Grade: 1, 2, 3

Learner Outcomes:

Grade 1, No. 4
Compare 2-D shapes to parts of 3-D objects in the environment.
[C, CN, V]

Grade 2, No. 9
Identify 2-D shapes as parts of 3-D objects in the environment.
[C, CN, R, V]

Grade 3, No. 6, 7
Describe 3-D objects according to the shape of faces and the number of edges and vertices.
[C, CN, PS, R, V]

Sort regular and irregular polygons according to the number of sides.
[C, CN, R, V]

Activity Description:

The purpose of this task is for the students to identify geometric shapes that they may see in their environment. It starts with groups of students looking for shapes, is followed by a class walk and the photographing of shapes, and culminates with children trying to identify where the "mystery" shapes are located from pictures that have been enlarged.

Background Information:

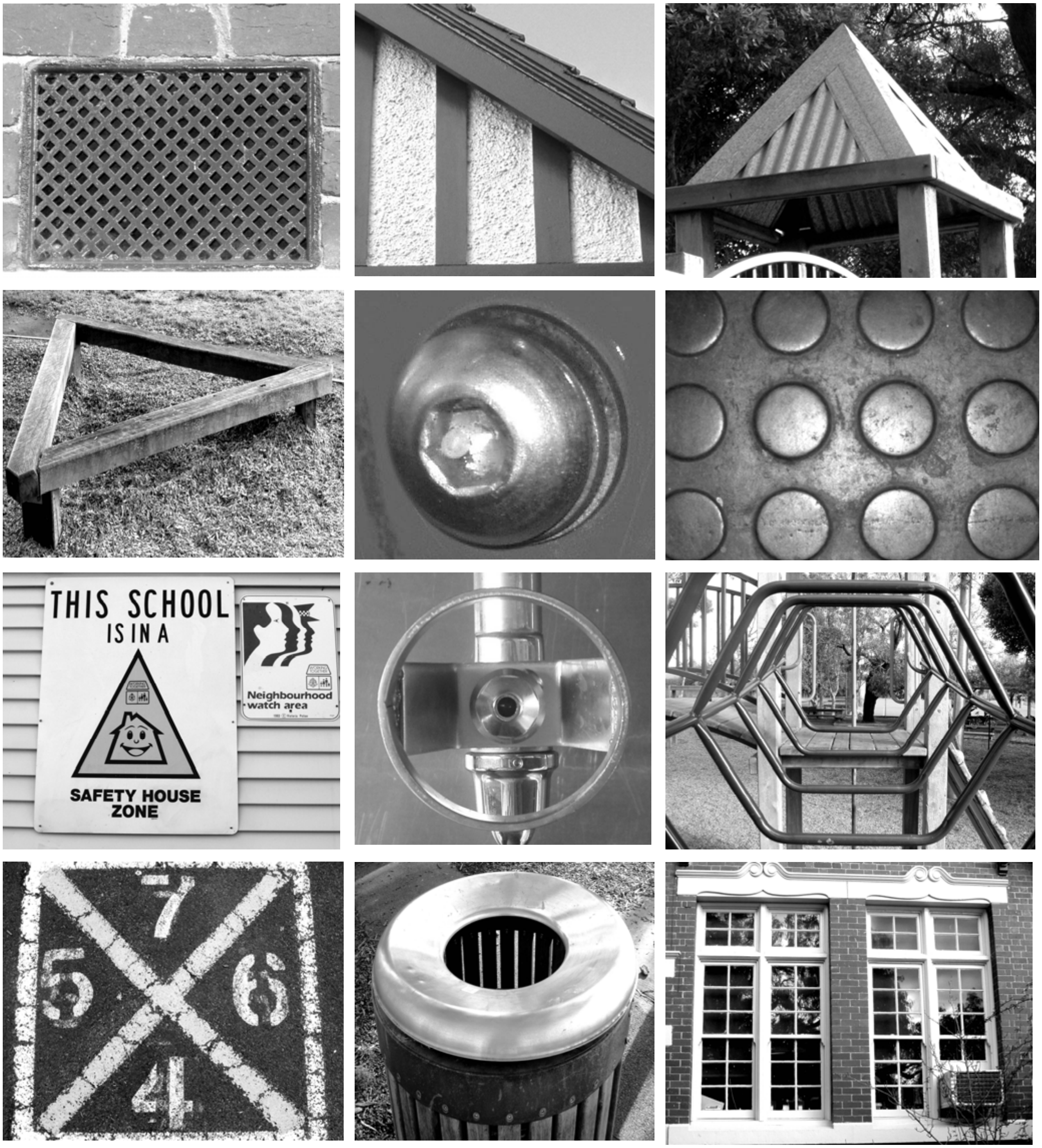
There is a difference between a triangle and a triangular shape. The triangular shape may have rounded corners while a triangle does not. For young students, this distinction is not as important but it should be discussed in grades 2 and 3 while doing tasks like this. The other aspect that is important to discuss is why these shapes are used. For example, why are the nuts on nuts and bolts often hexagonal? Why is a can of food cylindrical rather than cubical? Why are there often triangles in construction?

For the nuts, the thickness of the metal around the hole is greater than a square nut as the hexagon is closer to a circle. There would be a difficulty, however, with a circular nut as there would be no corner on which to gain purchase to tighten the nut.

A small experiment can assist with the question of the can. Using a piece of paper, make a rectangular prism shape. Fill it with rice or some other similar material. The sides will bulge out as they come under pressure. The other reason is that there is less metal used for more contents in a cylinder compared to a cube.
The triangles are used in construction for reasons of strength. This can be seen in the construction activity with straws.

The Activity:

1. This activity came from Faye's Grade 2 class. During the day, she sent small groups of three students with a parent volunteer and a small notebook in which to record their ideas outside the classroom to look for shapes in the environment of the school. Their task was to record what shapes they saw and where they saw them. Each group had a time limit and were on their honour to take responsibility.
2. After all groups had been out, Faye took the whole class on a walk and each group showed the shapes they had found. The students used the digital camera to photograph the item in its situation.
3. Faye downloaded the photographs, did a little editing, and then printed them out in black and white, putting them up on the wall in a display the following day.
4. The students then had the challenge of trying to identify where each shape was in the school. After the editing, the size of many objects was different and this added to the challenge.
5. Later they moved the display to the corridor and challenged all the students to find the shapes outside in the school or school grounds.
   - If you have a chance before the professional development session, it is a good idea to take a few photographs in the environment of the session, turn them to black and white, and challenge the participants to find the shape in the environment.
   - Some photographs taken on a school walk are shown here.
6. When the students have had a chance to look at all the pictures and to decide where they were in the school, consolidate the learning by drawing attention to each of the shapes and ask for the name and how they know it is that particular shape.
   - The discussion should include "How are the objects in the environment like the shapes found in the pattern blocks? How are they different?"
   - Ask "What made it difficult to match the shape in the photo to the real shape?"
   - Follow this with a discussion based on "What strategies did you use to match the shape in the photo to the real shape?"


THIS SCHOOL
IS IN A
SAFETY HOUSE
ZONE
Neighbourhood
watch area
7
5
6
4

Straw and String Constructions

Materials:

- Straws
- Cotton string
- Scissors
- or any of the following may be used:
  For edges: straws, toothpicks, bamboo skewers with the ends cut off
  For vertices: pipe cleaners, play dough, marshmallows.

Grade: 2, 3

Learner Outcomes:

Grade 2, No. 7
Describe, compare and construct 3-D objects.
[C, CN, R, V]

Grade 3, No. 6
Describe 3-D objects according to the shape of their faces and the number of edges and vertices.
[C, CN, PS, R, V]

Activity Description:

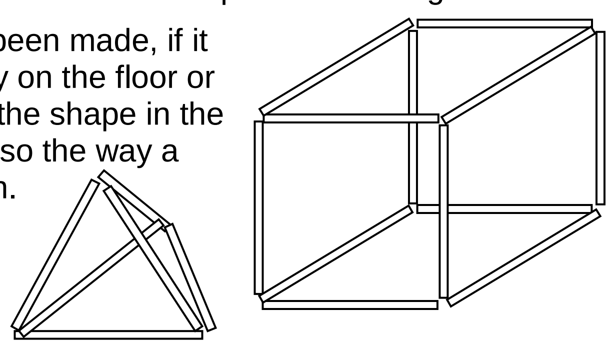
The purpose of this task is for the students to construct frame models of 3-D objects. In the process they will describe the models using the terms vertices and edges. They will also develop an understanding of these terms through this activity. In the process of doing this, they will discover the strength of the frame of a triangle in comparison to a rectangle.

Background Information:

Once the lengths of three sides of a triangle are set, there is only one triangle that can be made. The angles are also fixed. This is not true for quadrilaterals. A rectangle frame is not stable unless a triangle is fastened to it—children will have seen this with the struts on the frames for housing.

The tetrahedron or triangular pyramid has four triangular faces and is very stable. This is in contrast to making a cube and relates back to the stability of a frame model of a triangle compared to a frame model of a square or rectangle.

When a cube has been made, if it is dropped carefully on the floor or a table, it will form the shape in the diagram. This is also the way a cube is often drawn.



The other key aspect that making framework models draws out is the idea of edges. The framework highlights the idea of edges and enables the students to easily see how the shapes are constructed of edges and vertices where the edges or straws join. Once edges and vertices are identified, the shape of each face can be discussed.

The Activity:

1. Students work in small groups or pairs. The students are provided with straws and string (or other materials as suggested). Each pair of students is asked to build a skeletal model of one of the following: square, other rectangles and triangles, with at least one pair on each table making a triangle and at least one pair making a square.
   - Straws can be cut to make triangles with sides of different length and to make different rectangles. All shapes are made by threading the string down the centre of the straws then tying a knot to finish the shape.

2. Ask the groups to look at the shapes they have made and to discuss all the things that are the same about the shapes and all the things that are different. Have them sort the shapes into two groups according to some rule, then see if others on the table can work out the rule. For example, students may sort according to the number of sides. They might sort into shapes that had all the sides the same length and those that used different lengthed sides. Do it again using a different rule.

Note: One difference that is really important is that the triangles keep their shape while the squares and rectangles do not. Where is this fact widely used? Most students have seen the frames of houses when they are being built and may have noticed the struts on the angle to form the triangle and thus keep the frame stable.

3. In consolidating the learning at the end of the lesson, draw attention again to the number of edges and vertices in each shape. Ask what else they have learned. One point which the participants should raise is the fixed nature of the triangle in comparison to the square or rectangle.

Reflection Cube

Materials:

- Sheet of templates (see blackline master p. 78)
- Pens
- Scissors
- Adhesive tape and/or glue.

Grade: Teachers and Grade 2

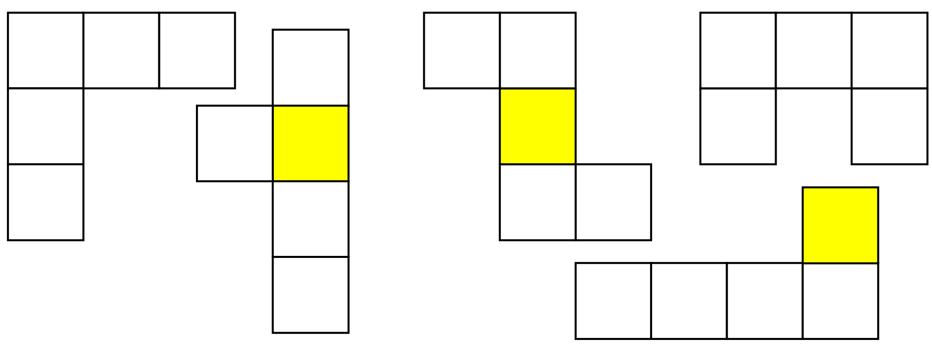
Learner Outcomes: Grade 2, No.8

Describe, compare and construct 2-D shapes including triangles, squares, rectangles and circles.

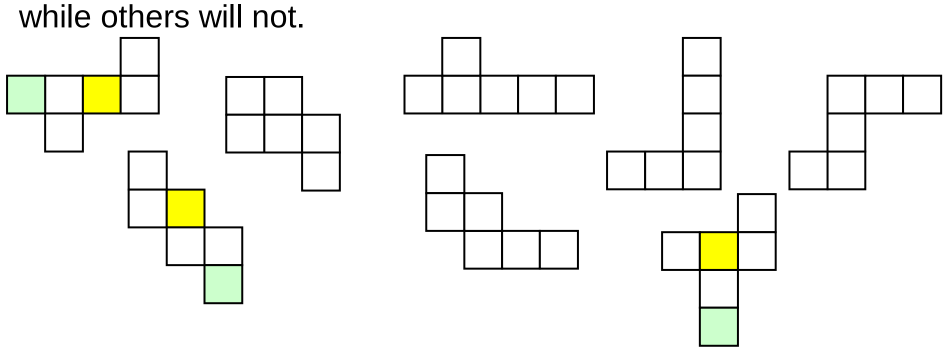
[C, CN, , R, V]

Activity Description: This activity serves two purposes. The first is to identify which nets would fold up into a cube and then to make the cube. The second purpose is a tool to encourage reflection on what has been learned.

Background Information: A pentomino is a set of five squares each joined to the others along a complete edge. There are 12 different pentominoes. Some will fold into the net of an open cube while others will not. Five of the twelve are shown here. Three of the five will fold into open boxes. The bottoms of these boxes are coloured.



If hexominoes, made of six squares so joined, are used then there are many more. Some of them will fold into closed boxes while others will not.



Of the eight hexominoes above, three will form complete boxes. On each of these three a possible base and top are coloured.

The Activity:

Activity for teachers

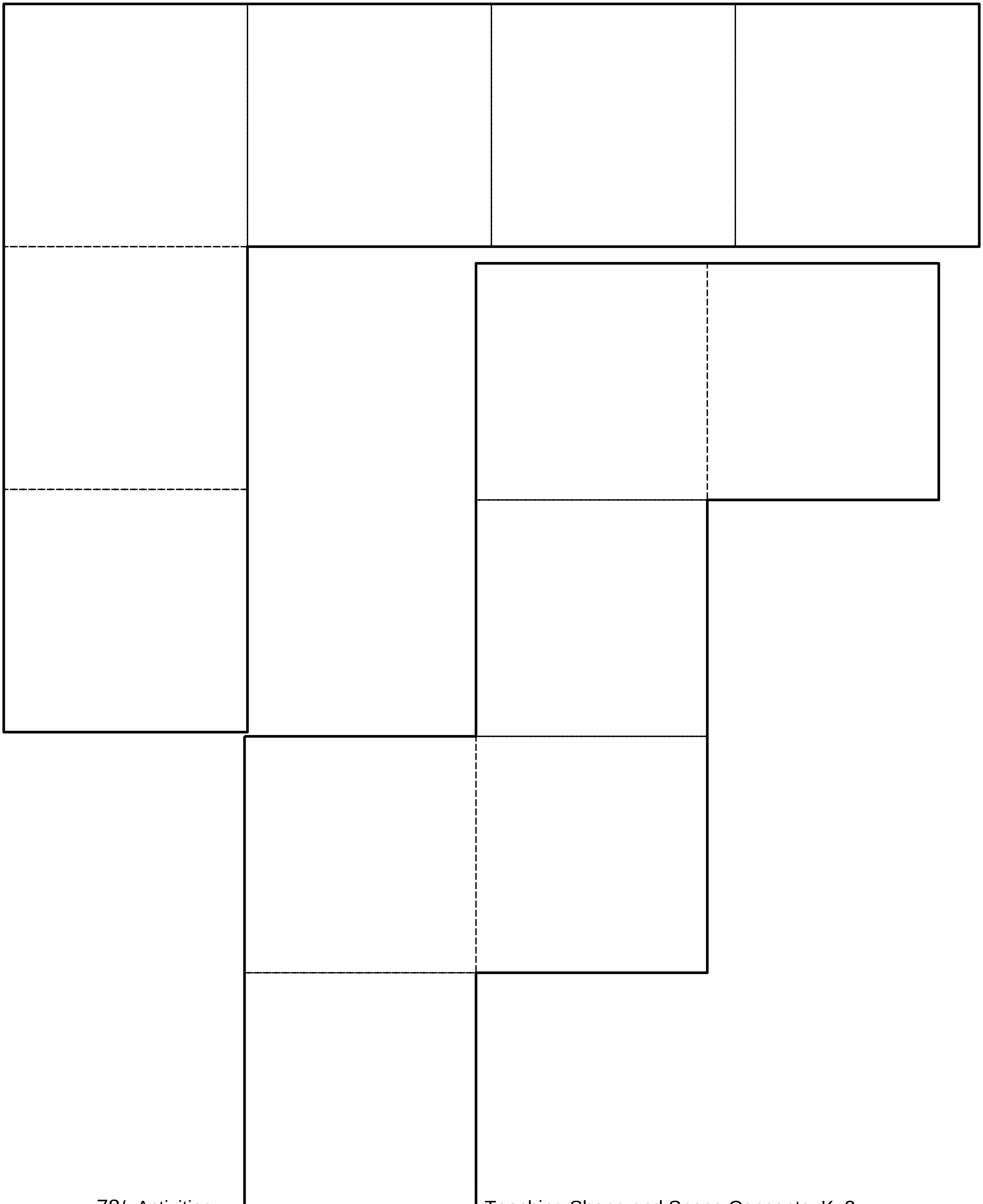
- After completing the discussion of big ideas of geometry and geometry teaching that have arisen during the day, participants are asked to choose one of the templates that they think will make a cube if it is cut out and folded.
- They are asked to choose six big ideas of geometry that they consider most important for them and their class, and to write each of these on one of the squares of the template. They are then asked to add flaps, cut out their cube and to fold it so that the ideas are on the outside.

Some of the big ideas that workshop participants might raise include:

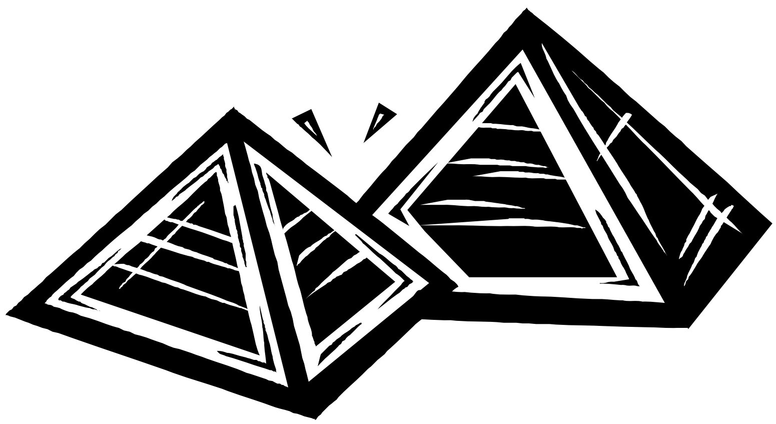
- the properties of shapes and how we identify shapes
- faces, edges and vertices of solids
- shapes in the environment and uses of them
- symmetry
- regular and irregular
- same and different
- construction of shapes (and stability).

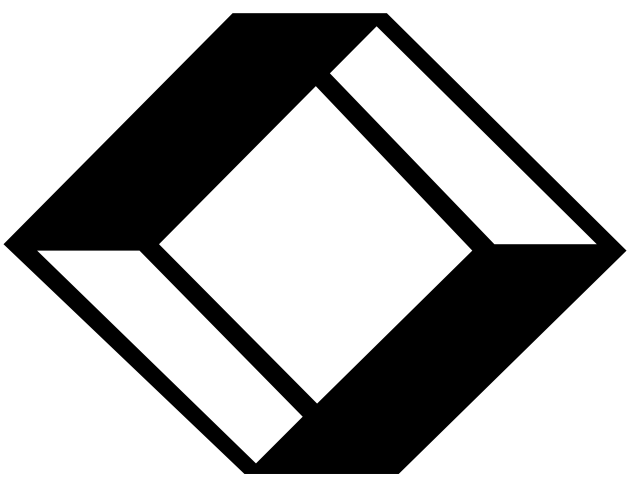
Activity for children

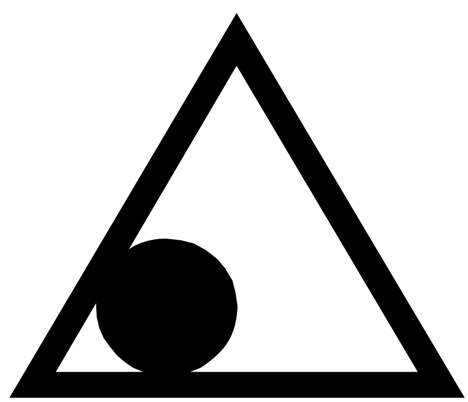
- The task should begin with pairs of students being given the challenge of trying to find all the possible pentominoes using squared paper or gluing squares on paper. Once they have found them, they are challenged to find which ones will fold into an open box.
- The hexominoes are then introduced. The students are asked to try to decide which ones will fold into a box before they cut them.
- Then ask the students to cut out the shapes. Ask them to fold the shapes carefully along the dotted lines. The final stage is to ask them which ones will make a cube.



Recording Sheets







WHAT IS MY SHAPE?/3-D CELEBRITY HEADS

Brief description: Students have a label with a shape on it attached to their back. They try to discover what their shape is by asking questions. When they know their shape, they move the sticky label to their front.

Rules: Move around and ask a different student each time.
Only ask questions which can be answered with Yes, No, I don't understand so please ask in another way, or I do not know how to answer that.
Once you know your shape, stick it on your front.

What is the mathematics?

What do we want students to notice?

Teacher role/questioning:

Features of the activity:

Adapt/extend/make simpler:

EXTRA NOTES:

CELEBRITY HEADS: A different version covering same ideas.

PROTOTYPES

Brief description: Students focus on describing and identifying 2-D regular and irregular polygons and images people have of shapes.

What is the mathematics?

What do we want students to notice?

Teacher role/questioning:

Features of the activity:

Adapt/extend/make simpler:

EXTRA NOTES:

MAKING SHAPES

Brief description: Students use triangles cut as half squares to create new shapes, then sort these shapes using a variety of sorting rules.

What is the mathematics?

What do we want students to notice?

Teacher role/questioning:

Features of the activity:

Adapt/extend/make simpler:

EXTRA NOTES:

FEELY BOX GEOMETRY

Brief description: Students use their hands to feel an unseen shape and determine its properties. They ask questions about a shape to try to identify it and draw it. The questions may only receive answers of yes; no; I don't understand so please ask it in another way; or please tell me how to find that out.

What is the mathematics?

What do we want children to notice?

Teacher role/questioning:

Features of the activity:

Adapt/extend/make simpler:

EXTRA NOTES:

CREATE A COPY

Brief description: Students work in pairs. One makes a shape using a small number of pattern blocks or 3-D blocks or other objects.

What is the mathematics?

What do we want students to notice?

Teacher role/questioning:

Features of the activity:

Adapt/extend/make simpler:

EXTRA NOTES:

QUICK DRAW

Brief description: This activity helps students develop their visualization skills. Students are shown a simple drawing for a short time interval on an overhead projector. The students are then asked to reproduce the drawing and share their strategy.

What is the mathematics?

What do we want students to notice?

Teacher role/questioning:

Features of the activity:

Adapt/extend/make simpler:

EXTRA NOTES:

USING TECHNOLOGY TOOLS

Brief description: Students use dynamic computer geometry software to explore the variety of triangles and other shapes.

What is the mathematics?

What do we want students to notice?

Teacher role/questioning:

Features of the activity:

Adapt/extend/make simpler:

EXTRA NOTES:

EXPLORING SHAPE AND OBJECTS

Brief description: Students sort items from a large collection and explain the attributes they are using.

What is the mathematics?

What do we want students to notice?

Teacher role/questioning:

Features of the activity:

Adapt/extend/make simpler:

Extra Notes:

SHAPES AROUND US

Brief description: Students explore their environment looking for the 2-D shapes. These are then photographed and used in a display with a challenge of recognizing where it is.

What is the mathematics?

What do we want students to notice?

Teacher role/questioning:

Features of the activity:

Adapt/extend/make simpler:

EXTRA NOTES:

STRAW AND STRING CONSTRUCTIONS

Brief description: Using cotton string passed down the centre of drinking straws, students build framework models of 3-D shapes after investigating triangles and squares.

What is the mathematics?

What do we want students to notice?

Teacher role/questioning:

Features of the activity:

Adapt/extend/make simpler:

EXTRA NOTES:

REFLECTION CUBE

Brief description: Participants choose a net that they think will make a cube. They reflect on things they have learned and choose six things to write on the faces of their cube. They then cut out the net and fold the cube.

What is the mathematics?

What do we want students to notice?

Teacher role/questioning:

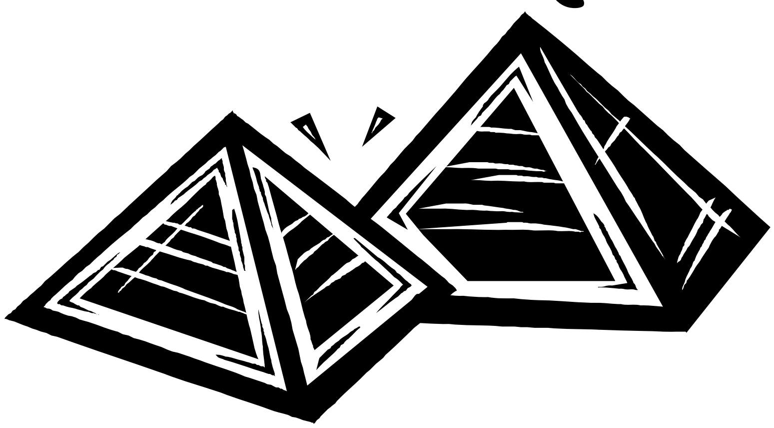
Features of the activity:

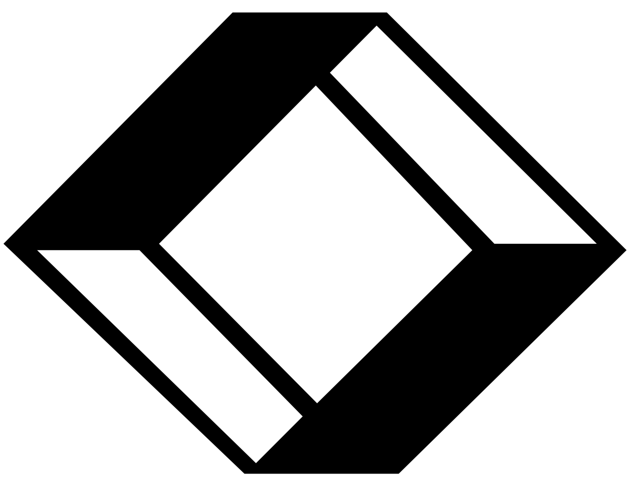
Adapt/extend/make simpler:

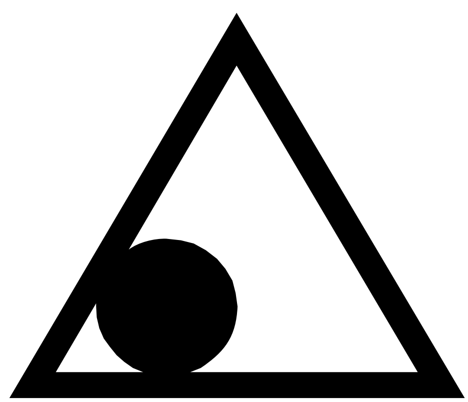
EXTRA NOTES:

Reflecting on the day:

Western and Northern Canadian Protocol Common Curriculum Framework for Mathematics (WNCP CCF)







[C]	Communication	**[PS]**	Problem Solving
[CN]	Connections	**[R]**	Reasoning
[ME]	Mental Mathematics and Estimation	**[T]**	Technology
		[V]	Visualization

Kindergarten **Strand:** Shape and Space (3-D Objects and 2-D Shapes)	**General Outcome:** Describe the characteristics of 3-D objects and 2-D shapes, and analyze the relationships among them.
Specific Outcomes *It is expected that students will:*	**Achievement Indicators** *The following set of indicators* ***may*** *be used to determine whether students have met the corresponding specific outcome.*
2. Sort 3-D objects using a single attribute. [C, CN, PS, R, V]	➢ Sort a given set of familiar 3-D objects using a single attribute, such as size or shape, and explain the sorting rule. ➢ Determine the difference between two given pre-sorted sets by explaining a sorting rule used to sort them.
3. Build and describe 3-D objects. [CN, PS, V]	➢ Create a representation of a given 3-D object using materials, such as modelling clay and building blocks, and compare the representation to the original 3-D object. ➢ Describe a given 3-D object using words, such as big, little, round, like a box and like a can.

[C]	Communication	**[PS]**	Problem Solving
[CN]	Connections	**[R]**	Reasoning
[ME]	Mental Mathematics and Estimation	**[T]**	Technology
		[V]	Visualization

Grade 1 **Strand:** Shape and Space (3-D Objects and 2-D Shapes)	**General Outcome:** Describe the characteristics of 3-D objects and 2-D shapes, and analyze the relationships among them.
Specific Outcomes *It is expected that students will:*	**Achievement Indicators** *The following set of indicators* ***may*** *be used to determine whether students have met the corresponding specific outcome.*
2. Sort 3-D objects and 2-D shapes using one attribute, and explain the sorting rule. [C, CN, R, V]	➢ Sort a given set of familiar 3-D objects or 2-D shapes using a given sorting rule. ➢ Sort a given set of familiar 3-D objects using a single attribute determined by the student and explain the sorting rule. ➢ Sort a given set of 2-D shapes using a single attribute determined by the student and explain the sorting rule. ➢ Determine the difference between two given pre-sorted sets of familiar 3-D objects or 2-D shapes and explain a possible sorting rule used to sort them.
3. Replicate composite 2-D shapes and 3-D objects. [CN, PS, V]	➢ Select 2-D shapes from a given set of 2-D shapes to reproduce a given composite 2-D shape. ➢ Select 3-D objects from a given set of 3-D objects to reproduce a given composite 3-D object. ➢ Predict and select the 2-D shapes used to produce a composite 2-D shape, and verify by deconstructing the composite shape. ➢ Predict and select the 3-D objects used to produce a composite 3-D object, and verify by deconstructing the composite object.
4. Compare 2-D shapes to parts of 3-D objects in the environment. [C, CN, V]	➢ Identify 3-D objects in the environment that have parts similar to a given 2-D shape.

[C]	Communication	**[PS]**	Problem Solving
[CN]	Connections	**[R]**	Reasoning
[ME]	Mental Mathematics and Estimation	**[T]**	Technology
		[V]	Visualization

Grade 2 **Strand:** Shape and Space (3-D Objects and 2-D Shapes)	**General Outcome:** Describe the characteristics of 3-D objects and 2-D shapes, and analyze the relationships among them.
Specific Outcomes *It is expected that students will:*	**Achievement Indicators** *The following set of indicators* ***may*** *be used to determine whether students have met the corresponding specific outcome.*
6. Sort 2-D shapes and 3-D objects using two attributes, and explain the sorting rule. [C, CN, R, V]	➢ Determine the differences between two given pre-sorted sets and explain the sorting rule. ➢ Identify and name two common attributes of items within a given sorted group. ➢ Sort a given set of 2-D shapes (regular and irregular) according to two attributes and explain the sorting rule. ➢ Sort a given set of 3-D objects according to two attributes and explain the sorting rule.
7. Describe, compare and construct 3-D objects, including: • cubes • spheres • cones • cylinders • pyramids. [C, CN, R, V]	➢ Sort a given set of 3-D objects and explain the sorting rule. ➢ Identify common attributes of cubes, spheres, cones, cylinders and pyramids from given sets of the same 3-D objects. ➢ Identify and describe given 3-D objects with different dimensions. ➢ Identify and describe given 3-D objects with different orientations. ➢ Create and describe a representation of a given 3-D object using materials such as modelling clay. ➢ Identify examples of cubes, spheres, cones, cylinders and pyramids found in the environment.

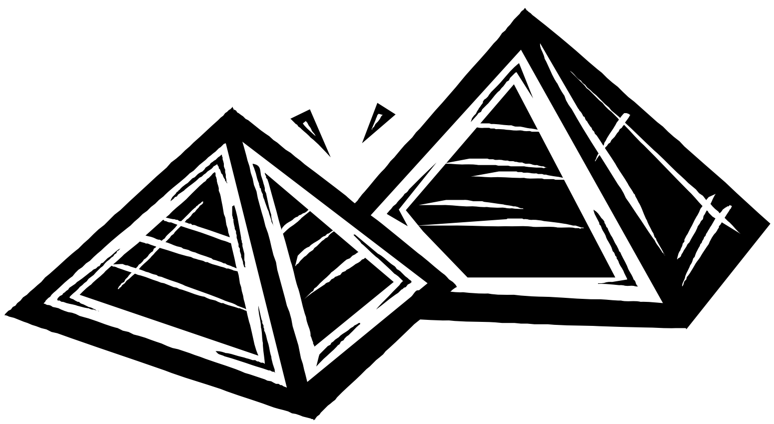
[C] Communication	**[PS]** Problem Solving
[CN] Connections	**[R]** Reasoning
[ME] Mental Mathematics and Estimation	**[T]** Technology
	[V] Visualization

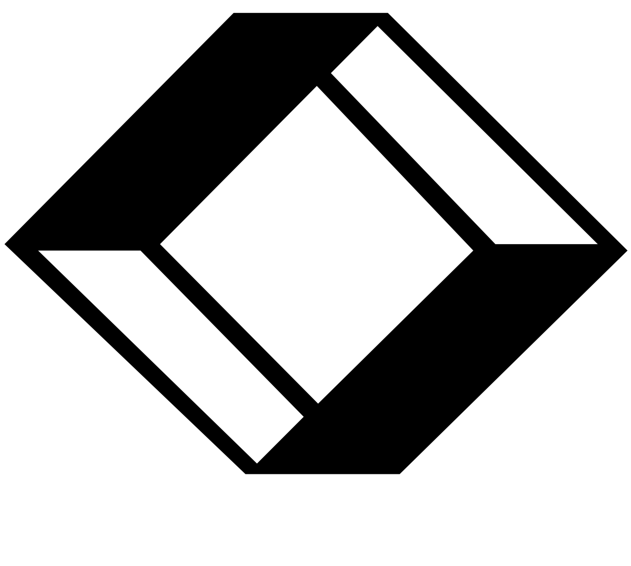
Grade 2 **Strand:** Shape and Space (3-D Objects and 2-D Shapes) (continued)	**General Outcome:** Describe the characteristics of 3-D objects and 2-D shapes, and analyze the relationships among them.
8. Describe, compare and construct 2-D shapes, including: • triangles • squares • rectangles • circles. [C, CN, R, V]	➢ Sort a given set of 2-D shapes and explain the sorting rule. ➢ Identify common attributes of triangles, squares, rectangles and circles from given sets of the same type of 2-D shapes. ➢ Identify given 2-D shapes with different dimensions. ➢ Identify given 2-D shapes with different orientations. ➢ Create a model to represent a given 2-D shape. ➢ Create a pictorial representation of a given 2-D shape.
9. Identify 2-D shapes as parts of 3-D objects in the environment. [C, CN, R, V]	➢ Compare and match a given 2-D shape, such as a triangle, square, rectangle or circle, to the faces of 3-D objects in the environment. ➢ Name the 2-D faces of a given 3-D object.

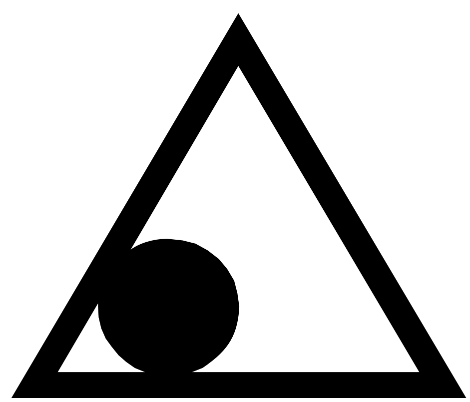
[C]	Communication	**[PS]**	Problem Solving
[CN]	Connections	**[R]**	Reasoning
[ME]	Mental Mathematics and Estimation	**[T]**	Technology
		[V]	Visualization

Grade 3 **Strand:** Shape and Space (3-D Objects and 2-D Shapes)	**General Outcome:** Describe the characteristics of 3-D objects and 2-D shapes, and analyze the relationships among them.
Specific Outcomes *It is expected that students will:*	**Achievement Indicators** *The following set of indicators* ***may*** *be used to determine whether students have met the corresponding specific outcome.*
6. Describe 3-D objects according to the shape of the faces, and the number of edges and vertices. [C, CN, PS, R, V]	➢ Identify the faces, edges and vertices of given 3-D objects, including cubes, spheres, cones, cylinders, pyramids and prisms. ➢ Identify the shape of the faces of a given 3-D object. ➢ Determine the number of faces, edges and vertices of a given 3-D object. ➢ Construct a skeleton of a given 3-D object and describe how the skeleton relates to the 3-D object. ➢ Sort a given set of 3-D objects according to the number of faces, edges or vertices.
7. Sort regular and irregular polygons, including: • triangles • quadrilaterals • pentagons • hexagons • octagons according to the number of sides. [C, CN, R, V]	➢ Classify a given set of regular and irregular polygons according to the number of sides. ➢ Identify given regular and irregular polygons having different dimensions. ➢ Identify given regular and irregular polygons having different orientations.

Research







SHAPING SHAPE IN SPACE

Marj Horne

Australian Catholic University

The development of understanding of the concepts of shape is an important part of the space strand of the curriculum. It requires an understanding of the properties of the different geometric shapes as well as curriculum ideas for working with children. The development of the background knowledge of shape as well as approaches to teaching are important in assisting children to a sound grasp of geometry.

Introduction

A couple of months ago I had occasion to be working with a group of teachers in a medium sized primary school on the topic Space. The teachers covered the full range from P - 6. Early in the session we used a simple activity which involved cutting a small nursery square of paper in two diagonally then making a shape with the two triangles so formed so that they joined along a complete edge. Fig 1 shows the types of shapes made.

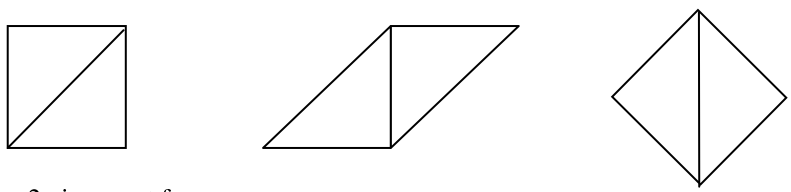


Figure 1. Shapes made from 2 pieces cut from a square.

In discussing the activity the teachers named the shapes they had in front of them. I was not expecting the exchange that followed. All were agreed that the first shape was a square. There was some discussion of the second shape. Was it a diamond? Perhaps it was a rhombus? What is a rhombus? (No answer). The decision was made that it was a parallelogram. The third shape caused even more discussion. It was a diamond. Was it a square? No – it was a diamond. No - it couldn't be a diamond because it was a square just turned around. Could a square also be a diamond? What exactly is a diamond? A baseball diamond is also a square – isn't it? During all the discussion the surprising thing was that no reference was really made to any of the properties of the shapes. There was no attempt made to define any of these shapes and even less common names such as rhombus were not related to properties.

This discussion disturbed me because it seemed that the teachers' knowledge base was not sound. What was their thinking? It seems to me that if we are going to teach children mathematics we need to understand the mathematics we are teaching and we also need knowledge of how children learn mathematics.

In the discussion the teachers were identifying shapes through recognition of a similarity to a visual image rather than through consideration of properties. The defining characteristic was that the shape looked like another rather than shapes being defined through consideration of properties. This approach can lead to a lack of clarity in thinking about shapes and to some of the common misunderstandings. For example in their discussion of diamond and square the only difference in the two shapes was the orientation. It was clear that the orientation of the shape affected the way in which it was seen and interpreted. Part of the problem arises from the use of common language terms such as diamond which carry particular images. The development of understanding of the shapes, their properties and the interrelations between them is something those teachers appeared to need in order to teach shape effectively.

Understanding shape

Van Hiele levels of understanding in Geometry can assist us to see some development in children's learning. The first three of these levels (0-2) are particularly relevant to children in primary school and level 3 is relevant to some and to their teachers. (For further levels and description see Clements & Battista (1992) and Horne (1998))

Level 0. **Precognition**. (not part of the van Hiele structure but added by Grouws (1992) as a pre-level 1). Shape is perceived but only with a subset of characteristics. For example, no differentiation between a square and a circle as both are closed shapes or differentiation between a square and a circle but not a square and a triangle by the characteristic of straight sides as opposed to curved sides.

Level 1. **Visual**. Shape is identified and operated on according to appearance. Identification is based on the congruency of visual properties. E.g., it is a cube because it looks like a box or it is a rectangle because it looks like a door.

Level 2. Descriptive/Analytic. Shapes are characterised by their properties. The focus is on relationships within classes rather than relationships between. A cube is now a cube because it is three dimensional with all faces the same sized squares.

Level 3. Abstract/Relational. Students are able to deal with abstract definitions understanding the idea of necessary and sufficient conditions, recognizing a hierarchy, and reasoning about the properties of classes of figures. Logical argument is part of this level.

In the example earlier the teachers in their discussion were demonstrating a level 1 understanding. They did not use the properties of the shapes and appeared unsure about the defining shapes by their characteristics. Young children's learning also show this stage. When a concept is introduced to young children it is usually through experiences showing them examples of that concept. If their experience is limited then the concept development is also restricted to one or two stereotypes examples forming the concept.

For example some young children involved in classifying shapes recently made a number of piles. The children placed the shapes we know as triangles in separate piles (shown in figure 2). The children referred to the first pile as triangles but they were unsure what to call the second pile. One child suggested half triangles (note 1). These children are classifying on the basis of "it looks like" but the images they have been shown are generally close to equilateral triangles. One child implied that there was not enough room inside for the thin ones to be triangles.

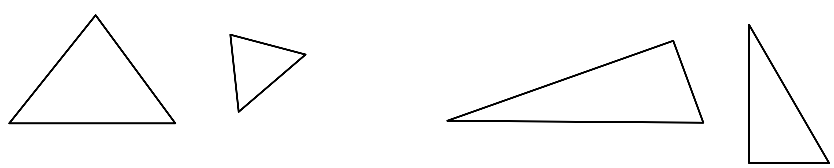


Figure 2. Two different groups of shapes.

In presenting triangles to children we need to be very clear about what a triangle is and ensure in our presentations that we include many different sizes, shapes and orientations as in figure 3. This is to avoid knowledge being based on just recognition of a limited number of stereotypes. Because most understanding of shape (and many other concepts) comes from experience it is important to experience both many different sizes and shapes of triangles and shapes which are not triangles. Examples of non-triangles include shapes with curved lines but 3 sides, shapes with more sides and non-closed shapes.

This may take place over a number of years as the concept is refined just as the concept of "red" is refined over the years.

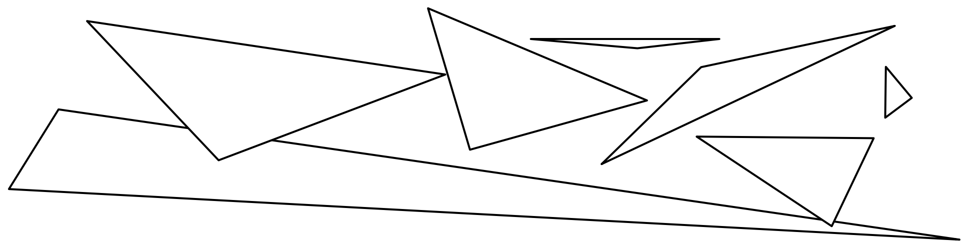


Figure 3. A range of triangleWorking on background knowledge.

At a later time classification occurs within the family of triangles as different types of triangles are introduced. This can give rise to a misunderstanding. While we classify triangles as isosceles, equilateral and scalene, these first two are not mutually exclusive. An **isosceles triangle** is a triangle with (at least) two sides the same length. **An equilateral triangle** has three sides the same length (It is also isosceles because it has two sides the same length). This means an equilateral triangle is a special case of isosceles triangles. A **scalene triangle** cannot be isosceles or equilateral as it has no sides the same. These classifications within the family of triangles should come much later in a child's experience than the original concept of triangle.

Figure 4 shows a diagrammatic representation of the family of triangles. Notice that right angled triangles can be either scalene or isosceles. The sizes of the circles and ellipses in the diagram are irrelevant.

The definitions above have been given in terms of lengths of sides. The angles can be considered properties. For example if a triangle has three equal sides (equilateral) it also has the property of three equal angles. As soon as one of these is known the other is automatically known. Similarly if a triangle has three equal angles it must have three equal sides. It also happens that these angles must be 60^o.

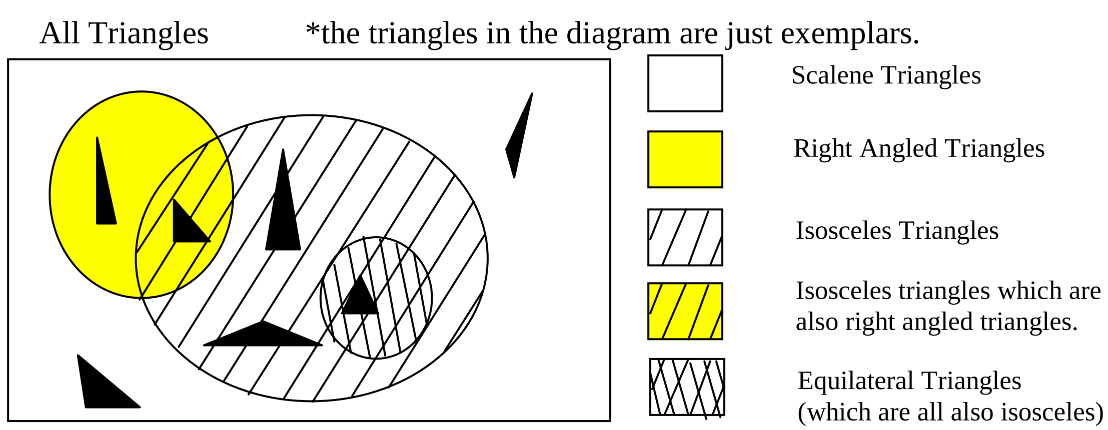


Figure 4. The family of triangles.

If a triangle has two equal sides it is isosceles and therefore must have two equal angles. Similarly if a triangle has two equal angles it must have two equal sides and so is isosceles.

Confusion about classification of shapes is more commonly noticed among the family of quadrilaterals. Many written worksheets with activities for young children incorrectly differentiate between a square and a rectangle. A square is, among other things, a special case of a rectangle. This will be further discussed after looking at the family of quadrilaterals.

A **quadrilateral** is a closed planar shape with four straight sides.

A **trapezium** is a quadrilateral with (at least) one pair of parallel sides. See Figure 5. (The arrows are to show parallel)

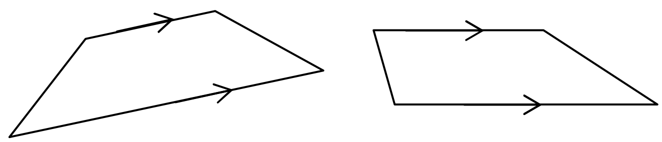


Figure 5. Two trapezia.

An **isosceles trapezium** has one pair of parallel sides with the other two sides equal in length but not parallel. This means it also has the angles on either one of the parallel sides equal as shown in Figure 6 (It is an isosceles triangle with top cut off parallel to the base). This means that the properties of an isosceles trapezium include two pairs of adjacent angles are equal and the diagonals are equal in length. (The same mark on two sides is meant to indicate same length. Similarly the same mark on angles indicates same size.)

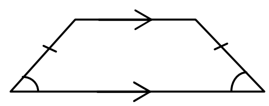


Figure 6. An isosceles trapezium.

A **parallelogram,** as shown in Figure 7, is a quadrilateral with two pairs of parallel sides. The two pairs of parallel sides gives the name parallelogram. Because of its parallel sides it also has the properties of opposite sides equal and opposite angles equal. Its diagonals are not necessarily of equal length but they bisect each other (cut each other in half).

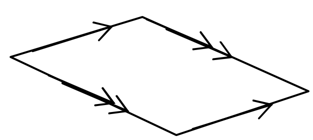


Figure 7. A parallelogram

One special case of a parallelogram arises when all four sides are equal and this produces a **rhombus**. To define a rhombus the minimum information is four equal sides. If a quadrilateral has four equal sides it must also have the properties of opposite sides parallel and opposite angles equal. Its diagonals bisect the corner angles, bisect each other and also bisect at right angles (but are not necessarily of equal length).

Sometimes this is called a diamond in common language but it should be presented with different orientations. A square is a special rhombus and notice that a baseball or softball diamond is actually a square. Diamond is not a mathematical term.

A **rectangle** can be defined in different ways. The name implies right angles and thus it can be defined as a quadrilateral with all right angles (note that it only needs to be known to have 3 right angles as if it has 3 it must have four). A rectangle is also a parallelogram as it has opposite sides parallel and hence has opposite sides of equal length. A rectangle has the property of its diagonals being of equal length as well as bisecting each other.

It is important to present rectangles with different orientations and many without horizontal and vertical lines. Figure 8 shows a group of rectangles of differing sizes and shapes with some showing non-horizontal and non-vertical lines

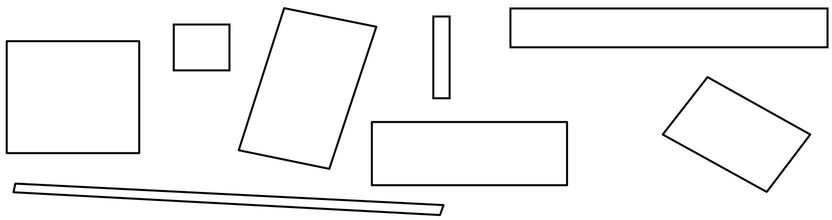


Figure 8. A family of rectangles

A **square** is both a special case of a rhombus and a special case of a rectangle. It is usually defined as having four equal sides and one right angle. (If it has one right angle and equal sides it must have four right angles). Its properties include those of the rhombus and the rectangle so its opposite sides are parallel, opposite angles equal (all are equal), diagonals are of equal length, bisect the corner angles and also bisect each other at right angles.

A **kite** is a quadrilateral that has two pairs of adjacent sides equal. Two often it is seen as only having one orientation and being more like the shape of the second kite in Figure 9.

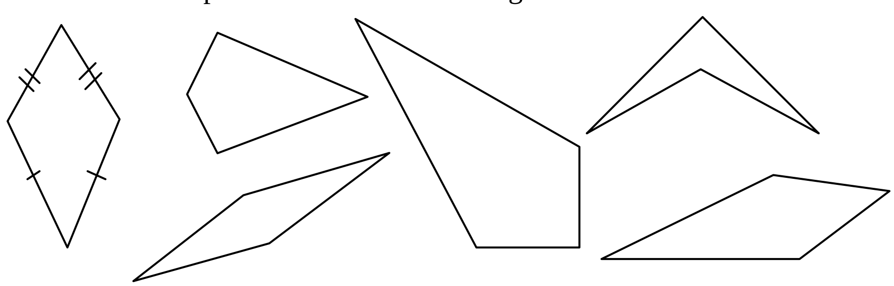


Figure 9. A family of kites

I have not put the equal length markers on all the diagrams as I think the equal sides show clearly. A rhombus is a special case of a kite as it has two pairs of adjacent sides equal (in fact the pairs are also equal to each other). Properties of kites are that the two angles not between the equal sides are equal, the diagonal joining the vertices between each of the equal sides bisects the angles and bisects the other diagonal at right angles (or its extension does in the case of the concave kite). This is shown in figure 10. A diagrammatic version showing how these quadrilaterals all fit together (a Venn diagram) is shown below in figure 11.

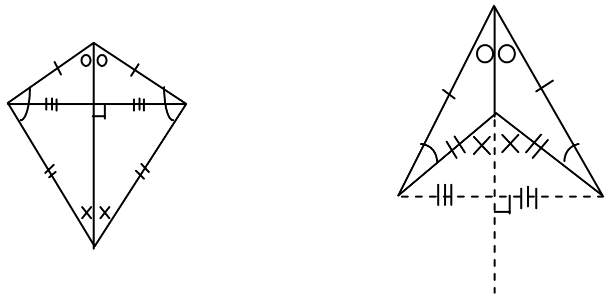


Figure 10. Two kites showing properties.

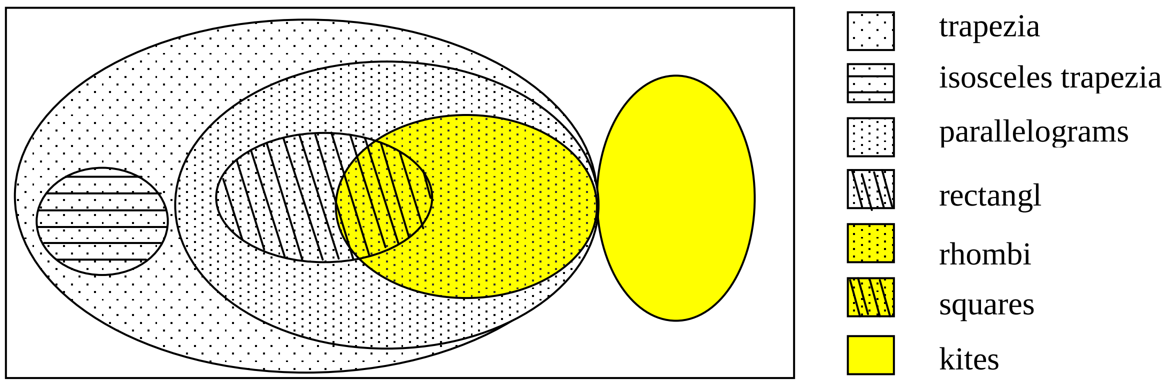


Figure 11. The family of quadrilaterals.

Figure 11 shows the growth in complexity of properties as well. The square has many more properties than the trapezium. Another aspect that needs to be stressed is the difference between properties and defining features. For example a rectangle is a quadrilateral with all its angles right angles. Other aspects can be regarded as properties. While they are true they are not part of the definition and can be shown to follow from the definition. However there are many sets of necessary and sufficient conditions for a rectangle. For example if a quadrilateral has its diagonals equal in length and bisecting each other, it must be a rectangle. Full understanding of the classification of shape comes with the understanding of the relationships between all the characteristics of the shapes and the relationships between the shapes themselves.

A final note

Teachers' knowledge affects the materials they present to children both in range and complexity. Without intending to they can thus limit the children they are teaching. The knowledge base above is not presenting material as it should be taught to children but rather the knowledge base teachers should have so that they can see where the curriculum is moving. The knowledge base should include not just the mathematics but also knowledge of children's learning in that area of curriculum. For shape the knowledge needs to go beyond the teacher being able to carry out the procedure of naming shapes to an understanding of the defining characteristics of different shapes, their properties, and the relationships between them including dynamic transformations moving from one shape to another.

This is supported by the recent work of Liping Ma (1999), who, in a comparison of American and Chinese teachers, strongly indicated that differences in the teachers' understanding of mathematics may play a critical role in differences in the children's development of mathematical concepts. Her work indicated that poorer results from American students could be caused by the poor understanding of mathematics shown by American teachers. Many showed only a procedural understanding of the mathematics they were teaching rather than a relational understanding.

In order to improve our students' understanding of mathematics we need to improve our own understanding and this is equally true in Space as it is in Number and Measurement.

References

Note 1. These reported children's responses are from a discussion with Jill Cheeseman and Barbara Clarke as a result of testing in the Early Numeracy Research Project.

Clements, D. H. & Battista, M. T. (1992). Geometry and spatial reasoning. In D. Grouws, (Ed.) *Handbook of research on mathematics teaching and learning* (pp. 420–464). New York: McMillan

Horne, M. (1998). An angle on geometry. In J, Gough and J. Mousley (Eds.) *Exploring all angles* (pp. 201–209). Brunswick, Victoria: Mathematical Association of Victoria.

Ma, L. (1999). *Knowing and teaching elementary mathematics*. Mahwah, New Jersey: Lawrence Erlbaum Associates.

Imagery and Mathematics Learning

The student who can construct and transform images is likely to be successful in doing mathematics.

There is compelling evidence that imagery plays a significant role in mathematical reasoning (Reynolds, 1993; Sfard, 1994). For example, a young child using a compensation strategy for adding 7 + 5 may think of "moving" one from the 7 to the 5 to form 6 + 6, a known fact. Or a child determining how many one inch cubes there are in a rectangular solid 3" by 3" by 4" may visualize the solid as composed of three layers. Whether working in a numerical or geometric context, when students are engaged in meaningful mathematics rather than rote computation, it is quite likely they will be using some form of imagery (Brown and Wheatley, 1989, 1991, 1994; Reynolds, 1993; Sfard, 1994). There is also compelling evidence that mathematicians use imagery in powerful ways (Hadamard, 1949; Nunokawa, 1994; Sfard, 1994). Mathematics is not just a logical subject but is laden with imagery. The word imaging is used as a metaphor for mental activity we cannot yet fully explain in terms of neural functioning.

While engaged in mathematical activity, whether of a numeric or geometric nature, students construct images. For example, they may be shown a geometric figure briefly and asked to draw what they saw. When they make their drawing, they are operating from a constructed image. The nature and quality of the image will influence the drawing which results. If at a later time they are asked to draw what they saw, the students then re-present the image.

When an image is constructed, only certain features of the experience are included (Kosslyn, 1983) . In Piagetian terms, the nature of the image results from the way the experience is assimilated. The nature of the image is dependent on prior mental constructions, intentions, and the situation under which the image is constructed. For example, an image of an isosceles triangle which has a horizontal base and a vertex angle above the base. If this image of a triangle is the child's only image of a triangle, then their concept of triangle is quite limited. Children have a richer concept of triangle when they can transform their image of triangle flexibly.

What an individual constructs depends on his or her mental schemes (images) formed from prior experiences. The line drawing shown in Figure 2 was presented briefly to a class and they were then asked to draw what they saw (Yackel and Wheatley, 1990). This figure was described as two squares, a small square and two trapezoids, a hallway, a skylight and a pyramid with the top cut off. Some individuals constructed an image of regions and others of joined segments, some two-dimensional and some three-dimensional interpretations. Even though the same figure was presented in the same manner to individuals, the nature of the images constructed varied greatly.

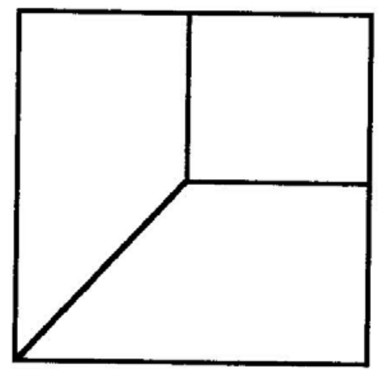


Figure 2. Quick Draw figure.

Learning from Students' Drawings

In designing activities for students, it is important to infer what the students know. Using our knowledge of students thinking we can choose tasks which are challenging but possible. Much can be learned about students' imagery and conceptualization by studying their drawings made in a Quick Draw setting. Their drawing activity provides a window into their minds. Students' activity is greatly influenced by their concepts and images. While we as adults may "see" a cube in the figure below, to some children it may be the shape made by three Pattern Blocks.

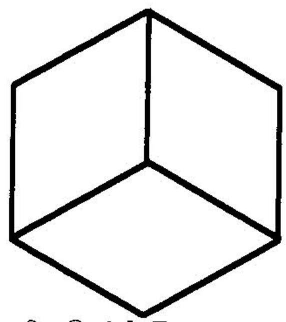


Fig. 3. Quick Draw shape.

Students do not just copy a figure using a photographic image of it. The image constructed is constrained by what the child knows. If a child's concept of triangle is a shape with a horizontal base and two segments slanting down to it, she may have difficulty drawing a right triangle.

Examples of students' actual drawings made in a Quick Draw setting are presented and discussed below. You are encouraged to analyze the drawings your students make.

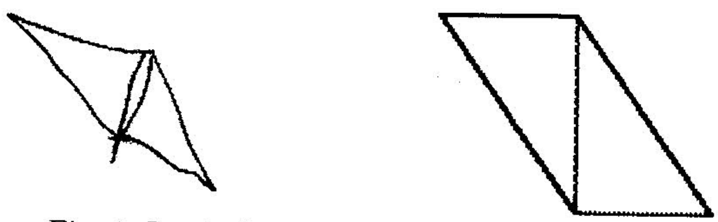


Fig. 4. Joan's drawing and QD shape shown.

Students draw what they "see." As shown in Figure 4 above, Joan drew two triangles because that is what she "saw" (mentally constructed). Rather than constructing an image of the whole figure, she partitioned it into two parts. It appears that Joan constructed the shape of a right triangle, since each of the triangles are nearly right triangles yet she did not construct the orientation of the triangles: there are no horizontal segments in her figure.

Figure 5. Mandy's drawing and QD shape shown.

It is very difficult for many primary grade students to draw a right triangle. Mandy drew two acute triangles with a common side (see Figure 5). There is no evidence in the drawing that she constructed any right angles, in fact, she most likely "saw" what we would call acute triangles.

Notice the path. She began at the top and drew the line down to the right, then the roughly horizontal base but, most strikingly, she continued up, slanting to the right, to complete her triangle. For her, triangles have sides slanting down to a horizontal base. The figure is completed by drawing two segments on the left. The top segment is not horizontal since again, she is drawing what she has conceptualized as a triangle.

This task has the potential of creating a perturbation, which can lead to a mental reorganization, a reorganization which may result in the conceptualization of a right triangle.

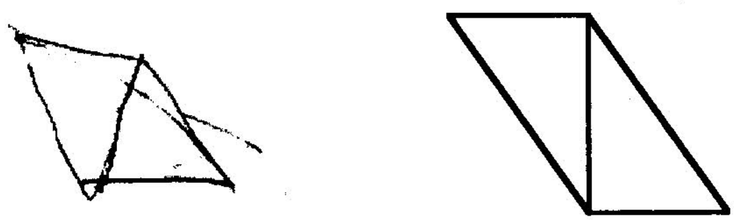
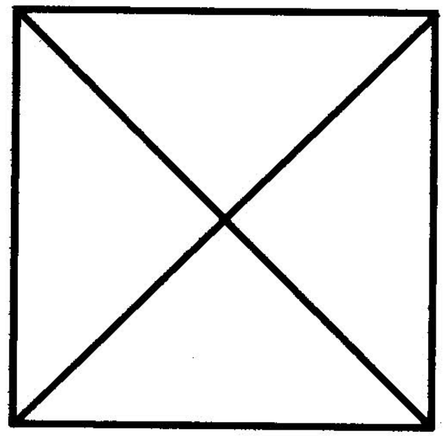
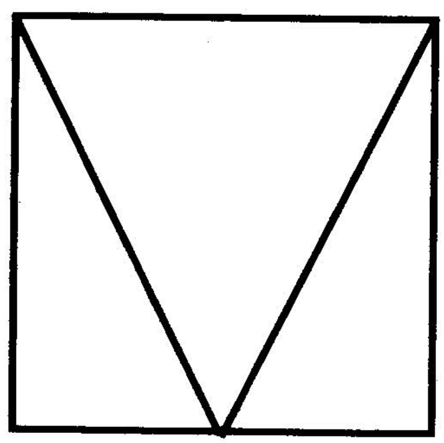
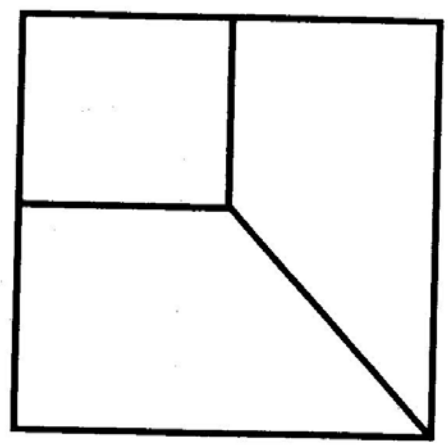


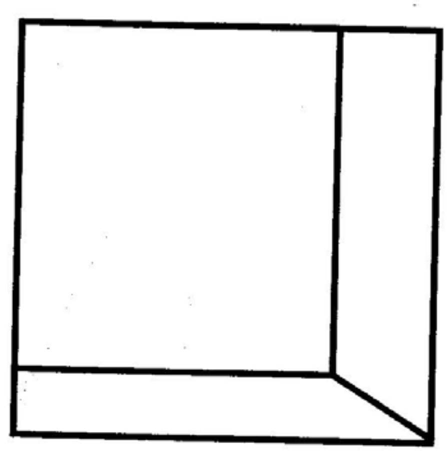
Fig. 6. David's drawing and QD shape shown.

The typical way of drawing a triangle with a horizontal base and two sides slanting down is quite dominating for many persons. David drew two acute triangles with no indication of any construction of right angles. The triangle on the right is nearly equilateral and likely is a depiction of his concept of triangle. The triangle formed on right is the "normal" orientation of a triangle. The top side of the triangle on the left is not horizontal as it was in the figure shown, perhaps because, for David, the triangle is "upside down." Also there was no attempt to form a vertical segment - triangles were constructed and the vertical orientation of this segment was not part of the mental construction.









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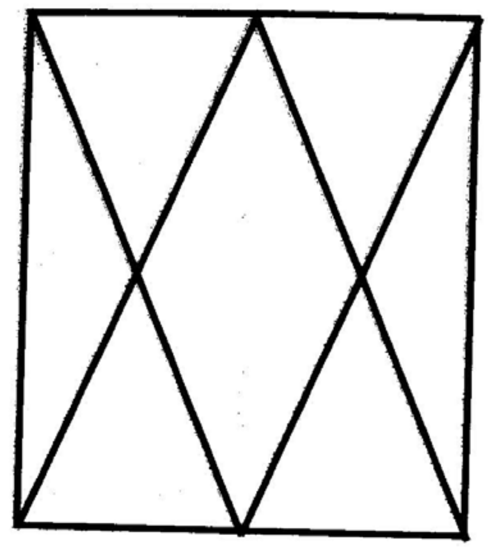
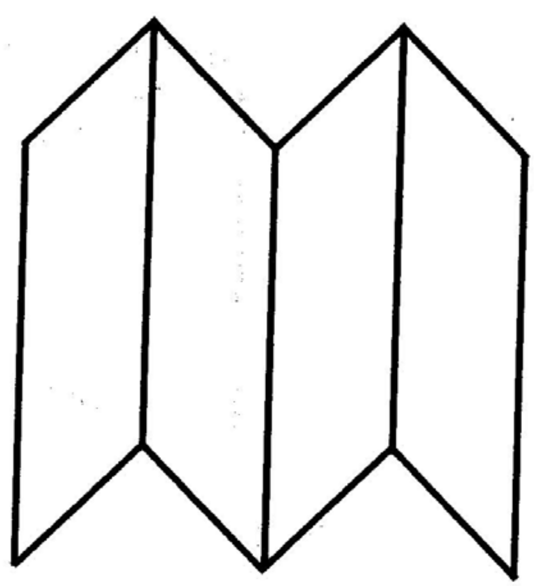




Quick Draw Directions

Quick Draw is designed to develop spatial sense, encourage the transformation of self-constructed images and develop geometric intuitions through discussion.

Quick Draw Should be used with the whole class. Overhead trans-parencies can be made from the blackline masters provided. Each pupil should have unlined paper and pencil. You may have them draw horizontal and vertical lines to divide the paper in fourths so they can make four drawings on one sheet. Show the firt design for about three seconds. Say, "I am going to show you a shape for three seconds and I want you to build a mental picture you can use to draw what you saw." Avoid the temptation to show it for a long period of time - - it is important that students work from imagery rather than copying what they are seeing. Say, "Draw what you saw." The students should have their pencils down during the presentation. At first you may get complaints about not having enough time. After a few moments show the shape again. Show again briefly if you feel it is necessary. This will only be necessary for more complex figures. Three times is unusually sufficient and two times is the norm. When students seem to be finished with their drawings, show the shape on the overhead so students can compare their drawing to the actual picture. With the design in view, ask "What did you see?" At times you may want to follow-up with "How did you draw it? What did you draw first?" This talking about mathematics encourages students to reflect on their imaging. Ask students to name the geometric figures they see. Geometric language will be used naturally. You may wish to supply mathematical name for such objects as trapezoids and parallelograms as needed by the students for communication. Much geometry can be learned through Quick Draw.

Follow the same procedure fo reach design. Present several shapes in a session. Be sure to use this activity initially for at least three days and then incorporate it with other topics throughout the year. This is an excellent activity with which to begin class. The figures in this book are organized in seven levels. There is a progression in complexity of the figures through the levels but there is considerable variability at each level. Some schools may wish to designate certain levels to be used at particular grade levels so that student swill not see figures they have drawn before. Select a design to present using your judgement of the appropriate level of complexity for your class. You will note broad individual differences in the students' initial drawing and descriptions.

As you use this activity, note the improvement in student's drawings and verbal descriptions. Also not individual differences. Some students may be very good at this task but not at arithmetic. Pay particular attention to the students who excel at this task since they may have unrealized mathematical potential.

Quick Draw *helps student develop*

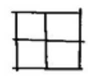
1. *metal imagery*
2. *recognition of shapes*
3. *analysis of mental images*
4. *spatial memory*
5. *concept of symmetry*
6. *geometric vocabulary*
7. *negotiate social norm*

QUICK DRAW

TEACHER MATERIALS AND PREPARATIONS:

1. Three or Four Quick Draw Transparencies.
2. Overhead projector (OH)
3. Blank paper for covering

STUDENT MATERIALS*:*
1. A pencil
2. unlined paper divided like this:



Whole class

"I will show you a shape for only a few seconds. Try to make a mental picture so you can draw it after I turn off the projector. Ready? On the count of three, ONE, TWO, THREE."
Turn on OH. Show the line pattern for 3 seconds. Turn off OH.
When most people have drawn all they can, prepare to show it to the students again.
"I will now give you another look. Ready?"
Show for 3 seconds.
When most people have drawn all they can, turn on the OH and leave it on.
Some students will be able to draw it looking at the shape when they could not draw it otherwise.

Whole class discussion

The discussion of their drawings is the heart of the activity.
"What did you see and how did you draw it?"
Encourage students to talk about their drawing. Do not rush the discussion.
Let it continue as long as new ideas are being put forth. Some students will be inspired by what others say. It is not unusual for five or more different ways of seeing the figure to be described.